# Stability in the Homology of Torelli Groups

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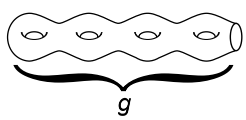
## Stability in the homology of Torelli groups

$\Sigma_{g,1}$ = compact orientable smooth genus-$g$ surface with 1 boundary component




Today's goal:

**Theorem (Miller–Patzt–Wilson)**

*Let $\mathcal{I}_{g,1}$ denote the Torelli group of $\Sigma_{g,1}$. The sequence of $Sp_{2g}(\mathbb{Z})$–reps $\{H_2(\mathcal{I}_{g,1};\mathbb{Z})\}_g$ is centrally stable for $g \geqslant 45$.*

Analogous results (Miller–Patzt–Wilson):
$\mathrm{IA}_n \subseteq \mathrm{Aut}(F_n)$, congruence subgroups of $\mathrm{GL}_n(R)$



Consider a compact orientable smooth surface $\Sigma_{g,1}$ with one boundary component. The goal of this talk is to motivate and explain the following result: the degree-2 homology groups $\{H_2(\mathcal{I}_{g,1};\mathbb{Z})\}_g$ of the Torelli group for this surface stabilize in a representation-theoretic sense.

In the same project we proved analogous results for the automorphism group of the free group, and certain congruence subgroups of general linear groups – but I will not discuss these results today.

## The Mapping Class Group

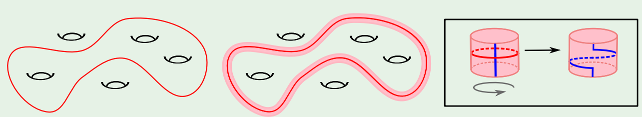
**Definition (Mapping Class Group $\mathrm{Mod}(\Sigma)$)**

Surface $\Sigma$.

$$\mathrm{Mod}(\Sigma) := \mathrm{Diffeo}^+(\Sigma, \partial\Sigma) \,/\, (\text{isotopy fixing } \partial\Sigma).$$

**Example (Dehn Twist about $\gamma$)**

$\gamma$ – simple closed curve in $\Sigma$



**Theorem (Dehn, Mumford, Lickorish, Humphries)**

*$\mathrm{Mod}(\Sigma_{g,1})$ is f.g. by $(2g+1)$ Dehn twists.*



Recall that for a surface $\Sigma$, the mapping class group of $\Sigma$ is the group of orientation-preserving diffeomorphisms that fix the boundary pointwise, up to smooth isotopy.

Important examples of mapping classes are the Dehn twists. If $\Sigma$ is an annulus, then its mapping class group is the infinite cyclic group generated by the following mapping class: rotate the base of the annulus by $360^\circ$ while holding the top fixed in place.

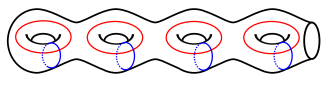
Given a simple closed curve $\gamma$ in any surface $\Sigma$, we can define the *Dehn twist $T_\gamma$ about $\gamma$* as follows. There is a neighbourhood of $\gamma$ that is homeomorphic to an annulus. We can apply the twist in this annulus, and extend the diffeomorphism to the rest of the surface by the identity.

Humphries proved that the mapping class group of a compact surface with 0 or 1 boundary components is finitely generated by the Dehn twists around an explicit set of $2g+1$ curves.

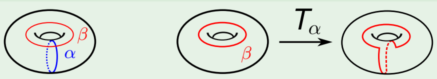
## Action on Homology

$\mathrm{Mod}(\Sigma_{g,1}) \circlearrowleft H_1(\Sigma_{g,1}, \mathbb{Z}) \cong \mathbb{Z}^{2g}$

$\rightsquigarrow \quad \mathrm{Mod}(\Sigma_{g,1}) \twoheadrightarrow \mathrm{Sp}_{2g}(\mathbb{Z})$



**Example (Closed Torus $T^2$)**




$T_\alpha(\beta) = \alpha + \beta$

$$\mathrm{Mod}(T^2) \xrightarrow{\cong} \mathrm{Sp}_2(\mathbb{Z}) \cong \mathrm{SL}_2(\mathbb{Z})$$

$$T_\alpha \longmapsto \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$T_\beta \longmapsto \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$



Since homotopic maps induce the same map on homology, there is an action of the mapping class group on the first homology of the surface. This first homology group has a symplectic structure defined by the intersection form, and so this action defines a representation from $\mathrm{Mod}(\Sigma_{g,1})$ to the symplectic group $\mathrm{Sp}_{2g}(\mathbb{Z})$. It is not too hard to verify that this map is surjective.

This action is something we can understand quite concretely. For example, if we consider a torus, then it is an exercise to check the following. The mapping class group is generated by the Dehn twists $T_\alpha$ and $T_\beta$. The twist $T_\alpha$ fixes the homology class of $\alpha$, and maps the homology class of $\alpha$ to the sum $\alpha + \beta$. Similarly, $T_\beta$ acts by the matrix shown. In this case, this representation is an isomorphism, and the mapping class group of the torus is $\mathrm{SL}_2(\mathbb{Z})$.

In a torus with one boundary component, then the kernel of this action is the free cyclic group $\mathbb{Z}$, generated by the Dehn twist about the boundary.

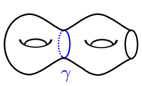
## The Torelli Group

**Definition (Torelli group $\mathcal{I}_{g,1}$)**

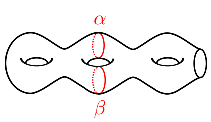
Torelli group $\mathcal{I}_{g,1}$ = kernel of the symplectic representation

$$1 \longrightarrow \mathcal{I}_{g,1} \longrightarrow \mathrm{Mod}(\Sigma_{g,1}) \longrightarrow \mathrm{Sp}_{2g}(\mathbb{Z}) \longrightarrow 1$$

Examples of mapping classes in $\mathcal{I}_{g,1}$:




separating curve
$T_\gamma \in \mathcal{I}_{g,1}$




homologous curves
$T_\alpha T_\beta^{-1} \in \mathcal{I}_{g,1}$



In general this action has a large kernel, called the Torelli group.

Examples of elements in Torelli are Dehn twists around separating curves, or products of Dehn twists around homologous curves as shown. In fact, elements of this form generate the Torelli group.

The Torelli group plays a role in several areas of geometry and topology. For certain results in the study of mapping classes, because we can get so much leverage from the action on first homology, the difficulty lies in understanding elements of the Torelli group. The Torelli group arises in 3–manifold theory (eg, the construction of homology 3–spheres) and in Teichmuller theory (eg, the definition of the period mapping).

We have realized the mapping class group as an extension of the symplectic group by the Torelli group. The slogan in low-dimensional topology is that the symplectic group is the “linear” and “understandable” part of the mapping class group, whereas the Torelli group is the “dark and mysterious” part.

## Finitness Properties of Torelli

**Finiteness Properties of Torelli**

**Theorem** (McCullough–Miller). $\mathcal{I}_{2,1}$ is not f.g.

**Theorem** (Johnson). $\mathcal{I}_{g,1}$ is f.g. for $g \geqslant 3$.

**Major Open Question**. Is $\mathcal{I}_{g,1}$ finitely presentable for $g \geqslant 3$?



McCullough–Miller proved that in genus 2, the Torelli group is **not** finitely generated. In earlier work Johnson proved that in genus at least 3, the Torelli group is finitely generated.

A significant open question in geometric group theory – a question which is not resolved in this project – is whether the Torelli group is finitely presentable.

## Homology of the Torelli Group

**Finiteness Properties of Torelli**

**Open Question**. Which groups $H_i(\mathcal{I}_{g,1})$ are f.g.?

$H_i(\mathcal{I}_{g,1})$ – known **not** f.g. for certain $i$
[Mess, Johnson–Millson–Mess, Hain, Akita, Bestvina–Bux–Margalit]

Little is known about $H_2(\mathcal{I}_{g,1})$.



The homology of the Torelli group has been widely studied, but it is difficult and basic questions remain open.

For example, we know from the work of Johnson that its homology is finitely generated in degree 1. Work of Bestvina–Bux–Margalit implies that the homology is **not** finitely generated in its top degree, and by work of several authors we know that it is not finitely generated in some other sporadic cases with $i$ large relative to $g$.

The question of which homology groups are finitely generated is open in general, and very little is known about the degree-2 homology in particular. The degree-2 homology is of particular interest because of its connection to questions of finite presentability.

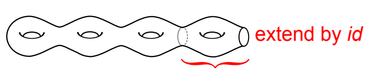
## Action on $\{H_*(\mathcal{I}_{g,1})\}_g$

**Key**: The sequence $\{H_2(\mathcal{I}_{g,1})\}_g$ has more structure.

$$1 \longrightarrow \mathcal{I}_{g,1} \longrightarrow \mathrm{Mod}(\Sigma_{g,1}) \longrightarrow \mathrm{Sp}_{2g}(\mathbb{Z}) \longrightarrow 1$$

$\rightsquigarrow \quad \mathrm{Sp}_{2g}(\mathbb{Z}) \circlearrowleft H_*(\mathcal{I}_{g,1})$.

$\Sigma_{g,1} \hookrightarrow \Sigma_{g+1,1}$




$\rightsquigarrow \quad \mathrm{Mod}(\Sigma_{g,1}) \to \mathrm{Mod}(\Sigma_{g+1,1})$ respects Torelli

$\rightsquigarrow \quad H_*(\mathcal{I}_{g,1}) \to H_*(\mathcal{I}_{g+1,1})$ $\mathrm{Sp}_{2g}(\mathbb{Z})$–equivariant



Fortunately, these homology groups have more structure.

The action of the mapping class group on the Torelli group by conjugation induces an action on its homology groups $H_*(\mathcal{I}_{g,1})$, which factors through an action by the symplectic groups $\mathrm{Sp}_{2g}(\mathbb{Z})$.

One of the general trends of representation stability is that, if we consider these short exact sequence of groups – given that the mapping class groups $\{\mathrm{Mod}(\Sigma_{g+1,1})\}_g$ are *homologically stable* – we expect that the homology of the kernels $\{\mathcal{I}_{g,1}\}_g$ will stabilize in some sense up to the action of the quotient groups $\mathrm{Sp}_{2g}(\mathbb{Z})$.

Moreover, because our surface has a boundary component, there are inclusions from the surface of genus $g$ to the surface of genus $g+1$. Hence a mapping class of $\Sigma_{g,1}$ acts on $\Sigma_{g+1,1}$ by extending from the genus $g$ subsurface by the identity. The induced maps of $H_2(\mathcal{I}_{g,1})$ are $\mathrm{Sp}_{2g}(\mathbb{Z})$–equivariant.

## $\{H_2(\mathcal{I}_{g,1})\}$ as an SI–module

**Key**: Realize $\{H_2(\mathcal{I}_{g,1})\}_g$ as a functor SI $\to$ AbGp.

Category SI (Putman–Sam)

objects = $\mathbb{Z}^{2g}$ with symplectic structure

morphisms = symplectic embeddings

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z}^2 & \longrightarrow & \mathbb{Z}^4 & \longrightarrow & \mathbb{Z}^6 & \longrightarrow & \cdots \\ & & \circlearrowleft & & \circlearrowleft & & \circlearrowleft & & \\ & & \mathrm{Sp}_2(\mathbb{Z}) & & \mathrm{Sp}_4(\mathbb{Z}) & & \mathrm{Sp}_6(\mathbb{Z}) & & \end{array}$$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H_2(\mathcal{I}_{1,1}) & \longrightarrow & H_2(\mathcal{I}_{2,1}) & \longrightarrow & H_2(\mathcal{I}_{3,1}) & \longrightarrow & \cdots \\ & & \circlearrowleft & & \circlearrowleft & & \circlearrowleft & & \\ & & \mathrm{Sp}_2(\mathbb{Z}) & & \mathrm{Sp}_4(\mathbb{Z}) & & \mathrm{Sp}_6(\mathbb{Z}) & & \end{array}$$



The key to the "representation stability" framework is to realize our sequence of homology groups as a module over the category SI, a category designed to encode the $\mathrm{Sp}_{2g}(\mathbb{Z})$–actions and the equivariant maps. This framework was developed largely in work of Church–Ellenberg–Farb and Putman–Sam.

The objects of SI are finite-rank free abelian group $\mathbb{Z}^{2g}$ with a symplectic structure, and the morphisms are symplectic embeddings. A skeleton of the category is shown. Notably, the endomorphisms are the symplectic groups $\mathrm{Sp}_{2g}(\mathbb{Z})$.

A module over this category will therefore be a sequence of abelian groups with $\mathrm{Sp}_{2g}(\mathbb{Z})$–actions, and compatible maps. For each fixed homological degree $i$, the sequence of homology groups $\{H_i(\mathcal{I}_{g,1})\}_g$ has exactly this structure.

The advantage of this formalism is that, having realized our sequence of homology groups $\{H_2(\mathcal{I}_{g,1})\}_g$ as an object in this abelian functor category, we can attack it using all the tools of commutative algebra.

## Results: stability for $\{H_2(\mathcal{I}_{g,1})\}$

**Theorem (Boldsen–Hauge Dollerup)**

*For $g > 6$,*

$$Sp_{2g}(\mathbb{Z}) \cdot im\, H_2(\mathcal{I}_{g-1,1};\mathbb{Q}) = H_2(\mathcal{I}_{g,1};\mathbb{Q})$$

**Theorem (Miller–Patzt–Wilson)**

*$H_2(\mathcal{I}_{g,1};\mathbb{Z})$ is centrally stable as an SI–module in degree $\leqslant 45$.*



Boldsen and Hauge Dollerup had previously shown that, for $g > 6$, the rational homology group $H_2(\mathcal{I}_{g,1};\mathbb{Q})$ is generated as an $\mathrm{Sp}_{2g}(\mathbb{Z})$–representation by the image of the group $H_2(\mathcal{I}_{6,1};\mathbb{Q})$ in genus 6.

They did not use this language, but we would say that this is the statement that, as an SI–module, the sequence of rational homology groups is generated in degree $\leqslant 6$.

We sought to improve this result in a two ways: we prove a result for the integral homology groups, and we promote the "finite generation" result to a "finite presentation" result. Our main theorem is *central stability* for the sequence of homology groups $\{H_2(\mathcal{I}_{g,1};\mathbb{Z})\}$ as an SI–module, with stable range $g > 45$. (Our result proves a generation degree worse than 6, so it does not imply the theorem of Boldsen–Hauge Dollerup).

## Consequences: stability for $\{H_2(\mathcal{I}_{g,1})\}$

**Corollary (Miller–Patzt–Wilson)**

*The sequence $\{H_2(\mathcal{I}_{g,1})\}_g$ is* **presentable** *as an SI–module in degree $\leqslant 45$.*

**Corollary (Miller–Patzt–Wilson)**

*The sequence $\{H_2(\mathcal{I}_{g,1})\}_g$ and all maps are determined by*

$$0 \longrightarrow H_2(\mathcal{I}_{1,1}) \longrightarrow H_2(\mathcal{I}_{2,1}) \longrightarrow \cdots \longrightarrow H_2(\mathcal{I}_{45,1})$$

**Corollary (Miller–Patzt–Wilson)**

*For $g > 45$, there is a partial resolution*

$$Ind^{Sp_{2g}(\mathbb{Z})}_{Sp_{2g-4}(\mathbb{Z})} H_2(\mathcal{I}_{g-2,1}) \longrightarrow Ind^{Sp_{2g}(\mathbb{Z})}_{Sp_{2g-2}(\mathbb{Z})} H_2(\mathcal{I}_{g-1,1}) \longrightarrow H_2(\mathcal{I}_{g,1}) \longrightarrow 0$$



Before I define "central stability", I will state some of its consequences.

Firstly, as an SI–module, the sequence has presentation degree $\leqslant 45$. This means that, in a sense, all the generators and relators of the homology are supported on subsurfaces of genus 45 and below.

Additionally, all the data of all the homology groups, the maps between them, and the $\mathrm{Sp}_{2g}(\mathbb{Z})$–structure, are completely determined by the first 45 terms in the sequence and the maps on these groups. (So, in principle, if we could compute the first 45 homology groups, we would know them all – though currently we are unable to compute the group $H_2(\mathcal{I}_{g,1})$ even in genus $g = 3$.) Again, this result implies the following structural result: the degree-2 homology of the Torelli group of a surface is determined by the action of mapping classes supported on subsurfaces of genus 45 or less.

In this stable range, there is an explicit presentation for the homology group $H_2(\mathcal{I}_{g,1})$. Notably, the group $H_2(\mathcal{I}_{g,1})$ only depends (in a relatively simple way) on the two previous terms in the sequence. This third corollary is in fact the definition of central stability.

## Proof Ingredients

- For an SI–module $\{V_g\}$, construct a chain complex

$$\cdots \longrightarrow \mathrm{Ind}^{\mathrm{Sp}_{2g}(\mathbb{Z})}_{\mathrm{Sp}_{2g-4}(\mathbb{Z})} V_{g-2} \longrightarrow \mathrm{Ind}^{\mathrm{Sp}_{2g}(\mathbb{Z})}_{\mathrm{Sp}_{2g-2}(\mathbb{Z})} V_{g-1} \longrightarrow V_g \longrightarrow 0$$

  **Main Lemma.** If $\{V_g\}$ is a *polynomial functor*, the homology satisfies a certain regularity result.

- **Theorem** (Hatcher–Vogtmann). The space of tethered chains in $\Sigma_{g,1}$ is $\left(\frac{g-3}{2}\right)$–connected.



Boldsen and Hauge Dollerup proved their results by using geometric group theory methods to study relations in the Torelli group. In contrast, we proved our results by studying commutative algebraic properties of the category of SI–modules.

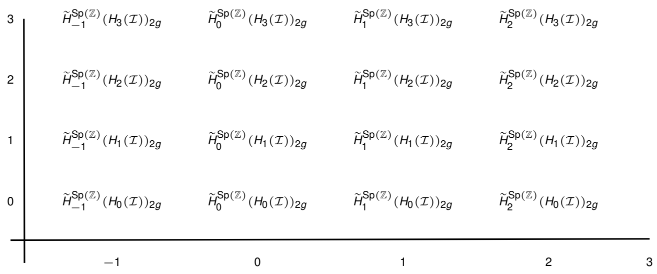
The presentation on the previous slide is in fact the tail of a chain complex of SI–modules, and it is a presentation for $g > 45$ because its homology groups vanish in this range. Call these homology groups *functor homology*.

The main technical lemma of our paper was the follwoing: using results of the second author Patzt we proved that, if $\{V_g\}$ belongs to certain class of well-behaved SI–modules called *polynomial functors*, then these functor homology groups vanish for $g$ large in a range depending only on the functor homology in degree 0 and 1. This lemma involves proving high connectivity for certain semi-simplicial complexes related to *complex of symplectic partial bases*.

We will also use a result of Hatcher–Vogtmann.

## Proof Ingredients

- spectral sequence analysis (Quillen homological stability argument)



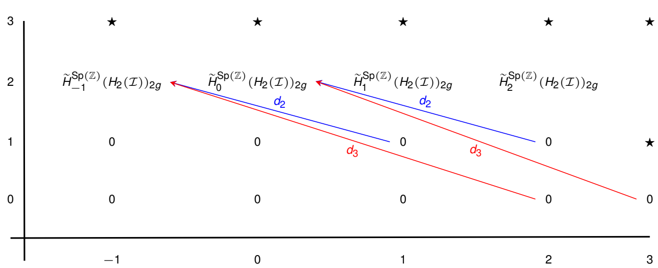



We can then prove the main theorem by analyzing a spectral sequence whose $E^2$ page encodes the functor homology groups of each of sequence of group homology groups of Torelli. Specifically, the $q^{th}$ row corresponds to the $q^{th}$ homology of Torelli, and the $p^{th}$ column corresponds to the the $p^{th}$ functor homology group.

The result of Hatcher–Vogtmann implies that this spectral sequence converges to zero in a range. Johnson's work computing $H_1(\mathcal{I}_{g,1})$ implies that this sequence of degree-1 homology groups form a polynomial SI–module, and therefore the first two rows of this spectral sequence vanish for $g$ sufficiently large.

## Proof Ingredients

- spectral sequence analysis (Quillen homological stability argument)






Since the spectral sequence converges to zero, it follows that for $g > 45$ the first two terms on row $q = 2$ must be zero. This immediately implies the partial resolution in the third corollary, and it implies the main theorem.

# Thank you!