Stability in the Homology of Torelli Groups

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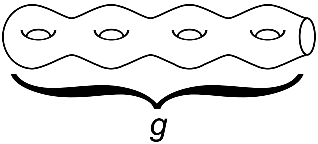
Stability in the homology of Torelli groups

$\Sigma_{g,1}$ = compact orientable smooth genus-g surface with 1 boundary component



Today's goal:

Theorem (Miller–Patzt–Wilson)

Let $\mathcal{I}_{g,1}$ denote the Torelli group of $\Sigma_{g,1}$. The sequence of $Sp_{2g}(\mathbb{Z})$–reps $\{H_2(\mathcal{I}_{g,1};\mathbb{Z})\}_g$ is centrally stable for $g \geqslant 45$.

Analogous results (Miller–Patzt–Wilson):
$\mathrm{IA}_n \subseteq \mathrm{Aut}(F_n)$, congruence subgroups of $\mathrm{GL}_n(R)$

The Mapping Class Group

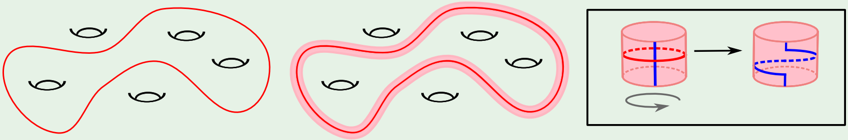
Definition (Mapping Class Group $\text{Mod}(\Sigma)$)

Surface Σ.

$$\text{Mod}(\Sigma) := \text{Diffeo}^+(\Sigma, \partial\Sigma) \text{ / (isotopy fixing } \partial\Sigma).$$

Example (Dehn Twist about γ)

γ – simple closed curve in Σ



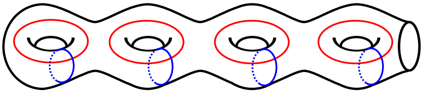
Theorem (Dehn, Mumford, Lickorish, Humphries)

$\text{Mod}(\Sigma_{g,1})$ is f.g. by $(2g+1)$ Dehn twists.

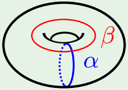
Action on Homology

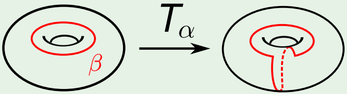
$\mathrm{Mod}(\Sigma_{g,1}) \circlearrowright H_1(\Sigma_{g,1}, \mathbb{Z}) \cong \mathbb{Z}^{2g}$

$\rightsquigarrow \quad \mathrm{Mod}(\Sigma_{g,1}) \twoheadrightarrow \mathrm{Sp}_{2g}(\mathbb{Z})$



Example (Closed Torus T^2)





$T_\alpha(\beta) = \alpha + \beta$

$$\mathrm{Mod}(T^2) \xrightarrow{\cong} \mathrm{Sp}_2(\mathbb{Z}) \cong \mathrm{SL}_2(\mathbb{Z})$$

$$T_\alpha \longmapsto \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$T_\beta \longmapsto \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

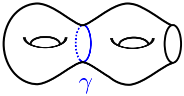
The Torelli Group

Definition (Torelli group $\mathcal{I}_{g,1}$)

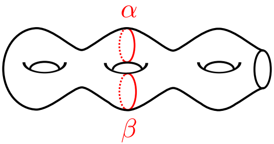
Torelli group $\mathcal{I}_{g,1}$ = kernel of the symplectic representation

$$1 \longrightarrow \mathcal{I}_{g,1} \longrightarrow \mathrm{Mod}(\Sigma_{g,1}) \longrightarrow \mathrm{Sp}_{2g}(\mathbb{Z}) \longrightarrow 1$$

Examples of mapping classes in $\mathcal{I}_{g,1}$:



separating curve
$T_\gamma \in \mathcal{I}_{g,1}$



homologous curves
$T_\alpha T_\beta^{-1} \in \mathcal{I}_{g,1}$

Finitness Properties of Torelli

Finiteness Properties of Torelli

Theorem (McCullough–Miller). $\mathcal{I}_{2,1}$ is not f.g.

Theorem (Johnson). $\mathcal{I}_{g,1}$ is f.g. for $g \geqslant 3$.

Major Open Question. Is $\mathcal{I}_{g,1}$ finitely presentable for $g \geqslant 3$?

Homology of the Torelli Group

Finiteness Properties of Torelli

Open Question. Which groups $H_i(\mathcal{I}_{g,1})$ are f.g.?

$H_i(\mathcal{I}_{g,1})$ – known **not** f.g. for certain i
[Mess, Johnson–Millson–Mess, Hain, Akita, Bestvina–Bux–Margalit]

Little is known about $H_2(\mathcal{I}_{g,1})$.

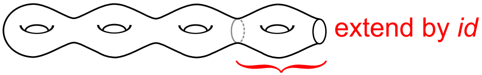
Action on $\{H_*(\mathcal{I}_{g,1})\}_g$

Key: The sequence $\{H_2(\mathcal{I}_{g,1})\}_g$ has more structure.

$$1 \longrightarrow \mathcal{I}_{g,1} \longrightarrow \mathrm{Mod}(\Sigma_{g,1}) \longrightarrow \mathrm{Sp}_{2g}(\mathbb{Z}) \longrightarrow 1$$

$\rightsquigarrow \quad \mathrm{Sp}_{2g}(\mathbb{Z}) \circlearrowleft H_*(\mathcal{I}_{g,1})$.

$\Sigma_{g,1} \hookrightarrow \Sigma_{g+1,1}$



$\rightsquigarrow \quad \mathrm{Mod}(\Sigma_{g,1}) \to \mathrm{Mod}(\Sigma_{g+1,1})$ respects Torelli

$\rightsquigarrow \quad H_*(\mathcal{I}_{g,1}) \to H_*(\mathcal{I}_{g+1,1})$ $\mathrm{Sp}_{2g}(\mathbb{Z})$–equivariant

$\{H_2(\mathcal{I}_{g,1})\}$ as an SI–module

Key: Realize $\{H_2(\mathcal{I}_{g,1})\}_g$ as a functor SI $\to$ AbGp.

Category SI (Putman–Sam)

objects = $\mathbb{Z}^{2g}$ with symplectic structure

morphisms = symplectic embeddings

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z}^2 & \longrightarrow & \mathbb{Z}^4 & \longrightarrow & \mathbb{Z}^6 & \longrightarrow & \cdots \\ & & \circlearrowright & & \circlearrowright & & \circlearrowright & & \\ & & \mathrm{Sp}_2(\mathbb{Z}) & & \mathrm{Sp}_4(\mathbb{Z}) & & \mathrm{Sp}_6(\mathbb{Z}) & & \end{array}$$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H_2(\mathcal{I}_{1,1}) & \longrightarrow & H_2(\mathcal{I}_{2,1}) & \longrightarrow & H_2(\mathcal{I}_{3,1}) & \longrightarrow & \cdots \\ & & \circlearrowright & & \circlearrowright & & \circlearrowright & & \\ & & \mathrm{Sp}_2(\mathbb{Z}) & & \mathrm{Sp}_4(\mathbb{Z}) & & \mathrm{Sp}_6(\mathbb{Z}) & & \end{array}$$

Results: stability for $\{H_2(\mathcal{I}_{g,1})\}$

Theorem (Boldsen–Hauge Dollerup)

For $g > 6$,

$$Sp_{2g}(\mathbb{Z}) \cdot im\, H_2(\mathcal{I}_{g-1,1};\mathbb{Q}) = H_2(\mathcal{I}_{g,1};\mathbb{Q})$$

Theorem (Miller–Patzt–Wilson)

$H_2(\mathcal{I}_{g,1};\mathbb{Z})$ *is centrally stable as an SI–module in degree* $\leqslant 45$.

Consequences: stability for $\{H_2(\mathcal{I}_{g,1})\}$

Corollary (Miller–Patzt–Wilson)

The sequence $\{H_2(\mathcal{I}_{g,1})\}_g$ *is* **presentable** *as an SI–module in degree* $\leqslant 45$.

Corollary (Miller–Patzt–Wilson)

The sequence $\{H_2(\mathcal{I}_{g,1})\}_g$ *and all maps are determined by*

$$0 \longrightarrow H_2(\mathcal{I}_{1,1}) \longrightarrow H_2(\mathcal{I}_{2,1}) \longrightarrow \cdots \longrightarrow H_2(\mathcal{I}_{45,1})$$

Corollary (Miller–Patzt–Wilson)

For $g > 45$*, there is a partial resolution*

$$Ind_{Sp_{2g-4}(\mathbb{Z})}^{Sp_{2g}(\mathbb{Z})} H_2(\mathcal{I}_{g-2,1}) \longrightarrow Ind_{Sp_{2g-2}(\mathbb{Z})}^{Sp_{2g}(\mathbb{Z})} H_2(\mathcal{I}_{g-1,1}) \longrightarrow H_2(\mathcal{I}_{g,1}) \longrightarrow 0$$

Proof Ingredients

- For an SI–module $\{V_g\}$, construct a chain complex

$$\cdots \longrightarrow \mathrm{Ind}_{\mathrm{Sp}_{2g-4}(\mathbb{Z})}^{\mathrm{Sp}_{2g}(\mathbb{Z})} V_{g-2} \longrightarrow \mathrm{Ind}_{\mathrm{Sp}_{2g-2}(\mathbb{Z})}^{\mathrm{Sp}_{2g}(\mathbb{Z})} V_{g-1} \longrightarrow V_g \longrightarrow 0$$

  Main Lemma. If $\{V_g\}$ is a *polynomial functor*, the homology satisfies a certain regularity result.

- **Theorem** (Hatcher–Vogtmann). The space of tethered chains in $\Sigma_{g,1}$ is $\left(\frac{g-3}{2}\right)$–connected.

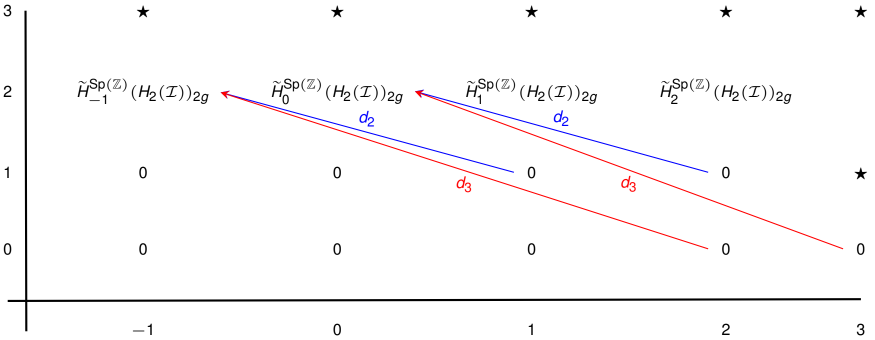
Proof Ingredients

- spectral sequence analysis (Quillen homological stability argument)

3	$\widetilde{H}^{\mathrm{Sp}(\mathbb{Z})}_{-1}(H_3(\mathcal{I}))_{2g}$	$\widetilde{H}^{\mathrm{Sp}(\mathbb{Z})}_{0}(H_3(\mathcal{I}))_{2g}$	$\widetilde{H}^{\mathrm{Sp}(\mathbb{Z})}_{1}(H_3(\mathcal{I}))_{2g}$	$\widetilde{H}^{\mathrm{Sp}(\mathbb{Z})}_{2}(H_3(\mathcal{I}))_{2g}$	
2	$\widetilde{H}^{\mathrm{Sp}(\mathbb{Z})}_{-1}(H_2(\mathcal{I}))_{2g}$	$\widetilde{H}^{\mathrm{Sp}(\mathbb{Z})}_{0}(H_2(\mathcal{I}))_{2g}$	$\widetilde{H}^{\mathrm{Sp}(\mathbb{Z})}_{1}(H_2(\mathcal{I}))_{2g}$	$\widetilde{H}^{\mathrm{Sp}(\mathbb{Z})}_{2}(H_2(\mathcal{I}))_{2g}$	
1	$\widetilde{H}^{\mathrm{Sp}(\mathbb{Z})}_{-1}(H_1(\mathcal{I}))_{2g}$	$\widetilde{H}^{\mathrm{Sp}(\mathbb{Z})}_{0}(H_1(\mathcal{I}))_{2g}$	$\widetilde{H}^{\mathrm{Sp}(\mathbb{Z})}_{1}(H_1(\mathcal{I}))_{2g}$	$\widetilde{H}^{\mathrm{Sp}(\mathbb{Z})}_{2}(H_1(\mathcal{I}))_{2g}$	
0	$\widetilde{H}^{\mathrm{Sp}(\mathbb{Z})}_{-1}(H_0(\mathcal{I}))_{2g}$	$\widetilde{H}^{\mathrm{Sp}(\mathbb{Z})}_{0}(H_0(\mathcal{I}))_{2g}$	$\widetilde{H}^{\mathrm{Sp}(\mathbb{Z})}_{1}(H_0(\mathcal{I}))_{2g}$	$\widetilde{H}^{\mathrm{Sp}(\mathbb{Z})}_{2}(H_0(\mathcal{I}))_{2g}$	
	-1	0	1	2	3

Proof Ingredients

- spectral sequence analysis (Quillen homological stability argument)



Thank you!