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A Gentle Introduction to Spectral Sequences:

The Leray-Serre Spectral Sequence.

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What is a spectral sequence?

A spectral sequence is a computational tool; they are more complex analogues of long exact sequences.

Eg Just as there is a LES of a pair (X,A) in homology / cohomology there is a spectral sequence associated to a filtration of subspaces $\emptyset \subseteq X_0 \subseteq X_1 \subseteq \cdots \subseteq X_n = X$.

Eg Analogous to the Mayer-Vietoris LES, there is a spectral sequence associated to an open cover $\{U_i\}$ of X.

The Structure of a spectral sequence.

A spectral sequence is a "book" consisting of a sequence of pages (or sheets), denoted E^r (homology) or E_r (cohomology) with $r \in \mathbb{N}$.

Each page has

- A 2D array of groups (or rings, or algebras) $E^r_{p,q}$, $(p,q) \in \mathbb{Z}^2$
- A map $d^r : E^r \longrightarrow E^r$ satisfying $(d^r)^2 = 0$, called the differential. Caution: r is an index, not an exponent.

The differentials d^r give E^r the structure of a chain complex.

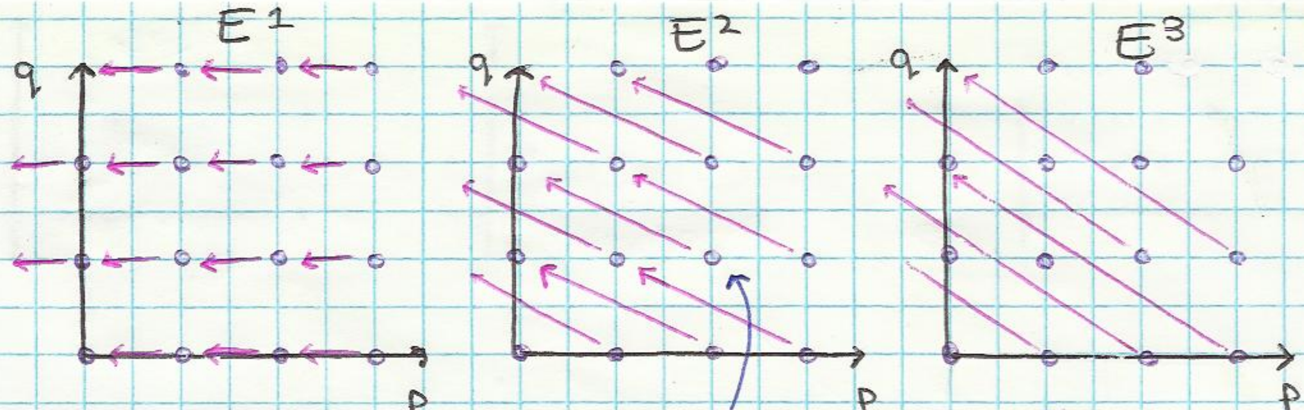
The page E^{r+1} is determined by the homology of E^r with respect to d^r

(though the differentials d^{r+1} are <u>not</u> determined by E^r, d^r).

Our main example will be the Leray-Serre Spectral sequence.

In the homology version:

- $E^r_{p,q}$ is non zero only for $p,q \geq 0$.
- $d^r_{p,q} : E^r_{p,q} \longrightarrow E^r_{p-r,\, q+r-1}$



- dots are abelian groups $E^r_{p,q}$
- differentials d^r

$d^1_{p,q} : E^1_{p,q} \to E^1_{p-1,q}$

Each abelian gp $E^2_{p,q}$ is a subquotient of the group $E^1_{p,q}$.

$d^2_{p,q} : E^2_{p,q} \to E^2_{p-2,q+1}$

$d^3_{p,q} : E^3_{p,q} \to E^3_{p-3,q+2}$

The sequence is 0 outside of the positive quadrant

At each stage, $E^r_{p,q}$ is replaced by a subquotient.
(Equivalently, each $E^r_{p,q}$ is a subquotient of $E^1_{p,q}$).

<u>Convergence</u>.

<u>Defn</u> We say a spectral sequence $\{E^r_{p,q}, d_r\}$ <u>converges</u> to a limit (denoted $E^\infty_{p,q}$) if, for some sufficiently large N (depending on p,q), all differentials leaving and entering $E^r_{p,q}$ are 0 for $r \geq N$.

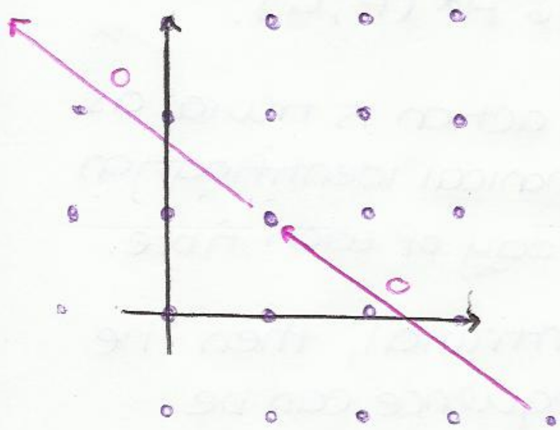
In this case, the homology groups are isomorphic for $r \geq N$:

$$E^N_{p,q} \cong E^{N+1}_{p,q} \cong E^{N+2}_{p,q} \cong \cdots$$

and these stable groups are called $E^\infty_{p,q}$.

<u>Defn</u> We say a spectral sequence degenerates at page N if $d^r \equiv 0$ for all $r \geq N$; in this case $E^N = E^\infty$.

Note: Since the Leray-Serre spectral sequence is 0 outside of the first quadrant, it must converge: For each (p,q), for r sufficiently large, the outgoing differential must eventually land outside the first quadrant, and the incoming differential must eventually originate outside the first quadrant.



<u>Eg</u> $E^3_{1,1} = E^\infty_{1,1}$

Notation: We write

$$E^r_{p,q} \Longrightarrow E^\infty_{p,q}$$

<u>The Leray-Serre Spectral Sequence.</u> (Homology version)

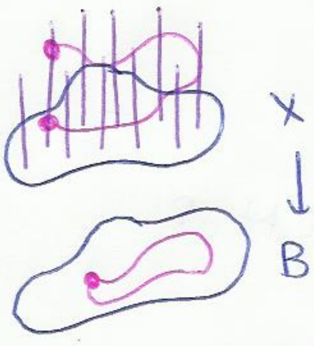
<u>Thm</u> Let $F \longrightarrow X \xrightarrow{\pi} B$ be a fibration; B path-connected.

Suppose $\pi_1(B)$ acts trivially on $H_*(F,G)$

Then there is a spectral sequence $\{E^r_{p,q}, d^r\}$ such that

- $d_r : E^r_{p,q} \longrightarrow E^r_{p-r,\,q+r-1}$, $\quad E^{r+1}_{p,q} = \dfrac{\ker(d^r)}{\operatorname{Im}(d^r)}$ at $E^r_{p,q}$
- $E^2_{p,q} \cong H_p(B; H_q(F,G))$. For G a field, $E^2_{p,q} = H_p(B) \otimes_G H_q(F)$
- Stable terms $E^\infty_{p,q}$ are isomorphic to successive quotients F^p_n / F^{p-1}_n in a filtration of $H_n(X;G)$: $(n = p+q)$

$$0 \leq F^0_n \leq F^1_n \leq \cdots \leq F^n_n = H_n(X;G).$$



Remarks (Action of $\pi_1(B)$ on $H_*(F;G)$).

Recall: Given a fibre $\pi^{-1}(b)$, a point $x \in \pi^{-1}(b)$ and a loop in B based at b,

we can lift the loop to a path from x to some $x' \in \pi^{-1}(b)$.

Thus the loop defines a map $\pi^{-1}(b) \longrightarrow \pi^{-1}(b)$.

Given an element $[\gamma] \in \pi_1(B,b)$ with rep γ, there is a map $\gamma : \pi^{-1}(b) \longrightarrow \pi^{-1}(b)$; since the fibration satisfies the homotopy lifting property, there is an induced action of $\pi_1(B,b)$ on $F \simeq \pi^{-1}(b)$ well-defined up to homotopy, and hence an induced action $\pi_1(B) \curvearrowright H^*(F;G)$.

We may think of the condition that this action is trivial as an "orientability condition", giving a consistent identification of the groups $H^n(F;G)$ with the cohomology of each fibre.

If the action $\pi_1(B) \curvearrowright H^*(F;G)$ is nontrivial, then the statement of the Leray-Serre spectral sequence can be formulated using homology with "twisted coefficients" the $\pi_1(B)$-module $H^*(F;G)$.

Remark (Cohomology version)

There is a cohomology version of the Leray-Serre spectral sequence of a fibration $F \longrightarrow X \longrightarrow B$. (under same assumptions)

$$\text{with } E_2^{p,q} = H^p(B; H^q(F;G)) \Longrightarrow H^{p+q}(X;G)$$

(again converging to successive quotients $E_\infty^{p,n-p} = F_p^n / F_{p+1}^n$ of $0 \subseteq F_n^n \subseteq \cdots \subseteq F_0^n = H^n(X;G)$)

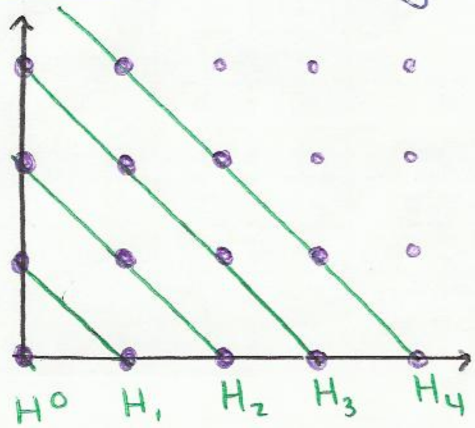
The differentials' directions are reversed: $d_r : E_r^{p,q} \longrightarrow E_r^{p+r,\, q-r+1}$ and $E_{r+1}^{p,q} = \ker(d_r) / \operatorname{Im}(d_r)$ at $E_r^{p,q}$.

<u>Remark</u> (the E^2 page)

At first encounter, the structure of the E^2 page may appear strange.

A remark, though: If we take $X = F \times B$, and we take the (co)homological version of the Leray-Serre spectral sequence over a field, then the sequence degenerates at E_2 and gives the Kunneth formula.

<u>Remark</u> (Recovering $H_n(X)$).



The sequence converges to successive quotients in a filtration of $H_*(X)$.

In general, this only determines $H_n(X)$ "up to extensions".

As an easy example, consider the short exact sequences:

$$0 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow ? \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

The extension ? could be either $\mathbb{Z}/4\mathbb{Z}$ or $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$

However, if we work with coefficients in a field, or if we have a spectral sequence converging to free $\mathbb{Z}$-modules, then the limit fully determines $H_n(X) = \bigoplus_p E^\infty_{p,n-p}$.

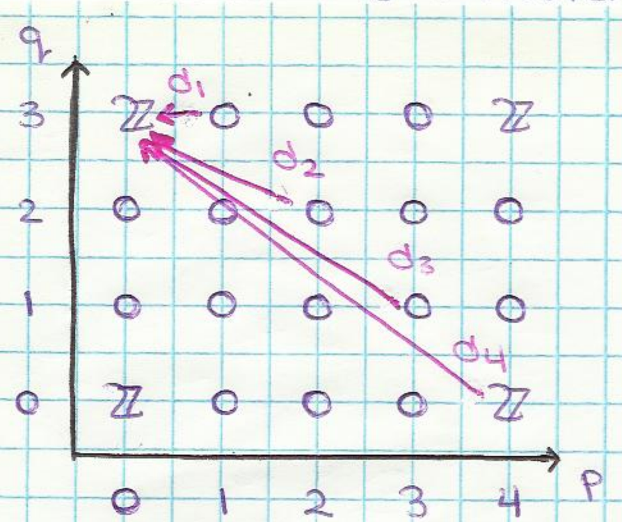
Generalized

Example: Hopf Fibration $S^3 \longrightarrow S^7 \longrightarrow S^4$

$$H_p(B;\mathbb{Z}) = H_p(S^4;\mathbb{Z}) = \begin{cases} \mathbb{Z}, & p = 0,4 \\ 0 & \text{otherwise} \end{cases}$$

$$H_q(F;\mathbb{Z}) = H_q(S^3;\mathbb{Z}) = \begin{cases} \mathbb{Z}, & q = 0,3 \\ 0 & \text{otherwise.} \end{cases}$$

NB S^4 is simply connected, so monodromy is trivial.



- All differentials up to d_4 must be 0, since they have either domain or codomain 0.
- Since the sequence converges to a filtration of

$$H_n(X) = \begin{cases} \mathbb{Z}, & n = 0,7 \\ 0 & \text{otherwise} \end{cases}$$

the terms $E^2_{0,3}$ and $E^2_{4,0}$ must be killed off.

Thus $d_4 : E^2_{0,3} \longrightarrow E^2_{4,0}$, the only possible nonzero differential, must be an isomorphism killing off $E^2_{0,3}$ and $E^2_{4,0}$.

The sequence degenerates at E^5.

NB We can use the structure of this spectral sequence to exclude some possibilities for fibrations of the form:

$$S^F \longrightarrow S^X \longrightarrow S^B.$$

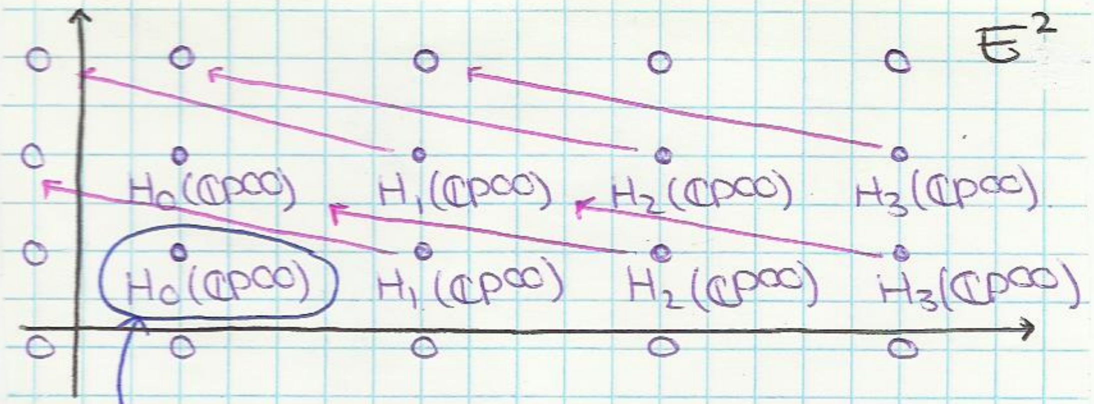
Example Using the Leray-Serre spectral sequence to compute $H_*(\mathbb{C}P^\infty;\mathbb{Z})$ from $H_*(S^1;\mathbb{Z})$ and $H_*(S^\infty;\mathbb{Z})$.

We have a fibration $S^1 \longrightarrow S^\infty \longrightarrow \mathbb{C}P^\infty$

Again, $\mathbb{C}P^\infty$ is simply connected.

so $E^2_{p,q} = H_p(\mathbb{C}P^\infty ; H_q(S^1;\mathbb{Z}))$

$$= \begin{cases} H_p(\mathbb{C}P^\infty;\mathbb{Z}), & q=0,1 \\ 0 & \text{otherwise.} \end{cases}$$

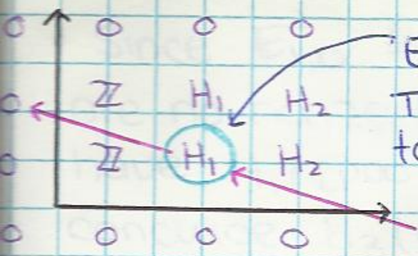


sequence converges to

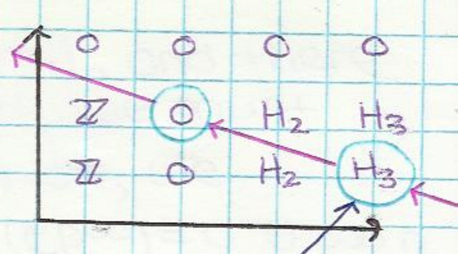
$$E^\infty_{0,0} = H_0(S^\infty;\mathbb{Z}) = \mathbb{Z}$$

and $E^\infty_{p,q} = 0$ for all other p,q since S^∞ is contractible.

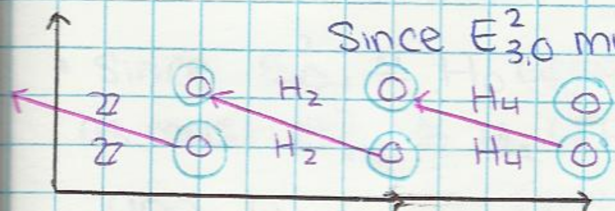
- The only possible nonzero differentiable is d^2, so $E^3 = E^\infty$.



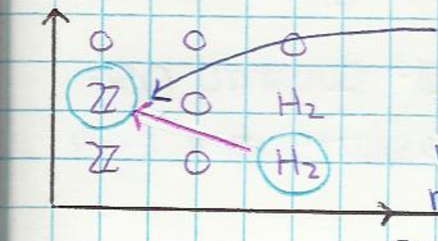
$E^2_{1,0} \cong H_1(\mathbb{C}P^\infty)$ must be 0 in limit. There are no nonzero differentials to kill this term, so we must have $H_1(\mathbb{C}P^\infty) \cong 0$.



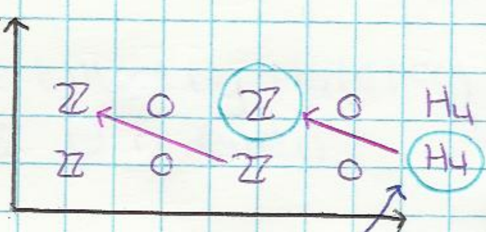
Since $E^2_{3,0}$ must die in the limit, we conclude $H_3(\mathbb{C}P^\infty)=0$



Iterating this argument, we conclude inductively that $H_n(\mathbb{C}P^\infty)=0 \ \forall\, n$ odd.



Since $E^2_{0,1} = H_0(\mathbb{C}P^\infty) \cong \mathbb{Z}$ must die in the limit, as must $E^2_{2,0} = H_2(\mathbb{C}P^\infty)$, we must have $H_2(\mathbb{C}P^\infty) \cong \mathbb{Z}$, and $d^2_{2,0} : H_2(\mathbb{C}P^\infty) \xrightarrow{\cong} H_0(\mathbb{C}P^\infty)$.



Then $E^2_{2,1} \cong H_2(\mathbb{C}P^\infty) \cong \mathbb{Z}$ must be killed off by $H_4(\mathbb{C}P^\infty)$ and continuing inductively $H_n(\mathbb{C}P^\infty) \cong \mathbb{Z}$ for all n even.