MSRI Graduate Summer School on Representation Stability
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Representation stability for configuration spaces of open manifolds

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These notes and exercises accompany a 2-part lecture series on representation stability results in configuration spaces of points in a manifold. Exercises are marked with an asterisk can be viewed as optional; these are either more advanced or are not necessary for the main goals of the worksheets.

Lecture 1: Configuration spaces and FI♯–modules

1 A review of configuration spaces

Last week, Andy Putman introduced configuration spaces.

1.1 Re-introducing configuration spaces

Definition I. (The (ordered) configuration space of a space M.) Let M be a topological space. Then the *(ordered) configuration space* of M on n points is the space of n-tuples of distinct points in M,

$$F_n(M) = \{(m_1, m_2, \dots, m_n) \in M^n \mid m_i \neq m_j \text{ for all } i \neq j\}$$

topologized as a subspace of M^n.

We can visualize points in $F_n(M)$ as in Figure 1.

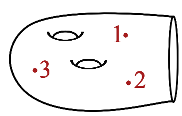


Figure 1: A point in $F_3(M)$ for an open surface M.

Exercise 1. (Connected components of configuration spaces).

(a) Let $I = (0, 1)$ denote the open unit interval. Illustrate the configuration spaces $F_1(I), F_2(I)$, and $F_3(I)$.

(b) Show that, for each $n \geq 0$, the space $F_n(I)$ has $n!$ connected components, and that each connected component is contractible.

(c) Let $J = (0, 1) \cup (2, 3)$ be the disjoint union of two open intervals. How many connected components does $F_n(J)$ have? Show that each is contractible.

(d) Let M be a connected manifold of dimension at least 2. Explain why, for each $n \geq 0$, the configuration space $F_n(M)$ is connected.

Exercise 2. (Configuration spaces of manifolds).

(a) Suppose that M is a manifold. Show that $F_n(M)$ is a manifold for all $n \geq 1$.

(b) If M is a d-dimensional manifold, then what is the dimension of $F_n(M)$?

Exercise 3. (Configuration spaces for small n). Prove the following.

(a) $F_1(M) = M$ for any topological space M.
(b) There is a deformation retract of $F_2(\mathbb{R}^d)$ onto S^{d-1}. In particular $F_2(\mathbb{C}) \simeq S^1$.
Hint: Consider the maps

$$F_2(\mathbb{R}^d) \longrightarrow S^{d-1} \qquad\qquad S^{d-1} \longrightarrow F_2(\mathbb{R}^d)$$

$$(x,y) \longmapsto \frac{(x-y)}{|x-y|} \qquad\qquad z \longmapsto (z,-z).$$

(c) There are homeomorphisms

$$F_n(\mathbb{R}^d) \cong \mathbb{R}^d \times F_{n-1}(\mathbb{R}^d \backslash \{0\})$$
$$F_n(\mathbb{C}^\times) \cong (\mathbb{C}^\times) \times F_{n-1}(\mathbb{C}^\times \backslash \{1\}).$$

Hint: Use the group structures on $\mathbb{R}^d$ and $\mathbb{C}^\times$.
Get stuck? See F. Cohen [Co, Example 2.6].

(d) Conclude from part (c) that for $n \geq 2$,

$$F_n(\mathbb{C}) \cong \mathbb{C} \times \mathbb{C}\backslash\{0\} \times F_{n-2}(\mathbb{C}\backslash\{0,1\})$$

Exercise 4. (Configuration spaces do not respect homotopy type). Show by example that even if M and M' are homotopy equivalent, then $F_n(M)$ and $F_n(M')$ need not be homotopy equivalent.

1.2 Unordered configuration spaces

The symmetric group S_n acts on $F_n(M)$ by permuting the n components of a point $(m_1, m_2, \ldots, m_n)$, equivalently, by permuting the labels on the n points as shown in Figure 1.

Exercise 5. (The S_n-action and its quotient).

(a) Show that S_n acts freely on $F_n(M)$. Conclude that the quotient map $F_n(M) \to F_n(M)/S_n$ is a covering space map.
(b) Show that the quotient $F_n(M)/S_n$ can be identified with the set of n-element subsets of M, and points can be visualized as in Figure 2.

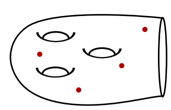


Figure 2: A point in $C_4(M)$ for an open surface M.

Definition II. (The unordered configuration space of a space M.) Let M be a topological space. Then the *unordered configuration space* $C_n(M)$ of M on n points is the quotient of $F_n(M)$ by the action of S_n. It is topologized as a quotient space.

Our goal for this lecture series is to study the homology of the configuration spaces of a non-compact manifold. To do this, we will introduce the following tool: FI♯–modules. Some of the following section will be a review of material from Andrew Snowden's lectures.

2 Induced FI–modules and FI♯–modules

2.1 Representable and induced FI–modules

For $n \in \mathbb{Z}_{\geq 1}$, let $[n] := \{1, , 2, \ldots, n\}$. Let $[0] := \varnothing$.

Exercise 6.

(a) Show that the endomorphisms $\mathrm{End}_{\mathsf{FI}}([n]) \cong S_n$ act on the set of morphisms $\mathrm{Hom}_{\mathsf{FI}}([m], [n])$ on the left by postcomposition, that is,

$$\begin{aligned} \sigma : \mathrm{Hom}_{\mathsf{FI}}([m], [n]) &\longmapsto \mathrm{Hom}_{\mathsf{FI}}([m], [n]) \\ \alpha &\longmapsto \sigma \circ \alpha \qquad\qquad \text{for all } \sigma : [n] \to [n] \end{aligned}$$

(b) Show that this action is transitive.

(c) Show that the stabilizer of the canonical inclusion $\iota_{m,n} : [m] \hookrightarrow [n]$ is

$$\{ \sigma \in S_n \mid \sigma \circ \iota_{m,n} = \iota_{m,n} \}$$

is isomorphic to S_{n-m}.

(d) Conclude that, as an S_n–set,

$$\mathrm{Hom}_{\mathsf{FI}}([m], [n]) \cong S_n / S_{n-m}.$$

Exercise 7.

(a) Show that the endomorphisms $\mathrm{End}_{\mathsf{FI}}([m]) \cong S_m$ act on the set of morphisms $\mathrm{Hom}_{\mathsf{FI}}([m], [n])$ on the right by precomposition, that is,

$$\begin{aligned} \sigma : \mathrm{Hom}_{\mathsf{FI}}([m], [n]) &\longmapsto \mathrm{Hom}_{\mathsf{FI}}([m], [n]) \\ \alpha &\longmapsto \alpha \circ \sigma \qquad\qquad \text{for all } \sigma : [m] \to [m] \end{aligned}$$

(b) Determine whether this action is transitive.

Let R be a commutative, unital ring. We will consider FI–modules over R, that is, functors from FI to the category of R–modules.

We know that any R–module is the quotient of a free R–module. We will see that the following special class of FI–modules $M(d)$ play the role of "free" FI–modules.

Definition III. (Representable FI–modules). Fix a nonnegative integer d. Define the FI–module $M(d)$ over R by

$$M(d)_n := R \cdot \mathrm{Hom}_{\mathsf{FI}}(d, n) \qquad \text{(the free } R\text{–module on the set } \mathrm{Hom}_{\mathsf{FI}}(d, n))$$

and the action of FI–morphisms by postcomposition. An FI–module of this form is called a *representable* FI–module.

Exercise 6 implies that the S_n–representation $M(d)_n$ is isomorphic to the coset representation $R[S_n/S_{n-d}]$.

Exercise 8.

(a) Show that $M(d)$ is generated by the identity morphism $\mathrm{id}_d \in M(d)_d$.

(b) Conclude that if $F : M(d) \to V$ is any map of FI–modules, then F is determined by $F(\mathrm{id}_d)$.

Exercise 9. Show that, as S_n–representations,

$$M(d)_n \cong \mathrm{Ind}_{S_{n-d}}^{S_n} R.$$

Exercise 10. Explicitly describe and compute the decompositions for the rational S_n–representations $M(0)_n$, $M(1)_n$, and $M(2)_n$.

Recall that the construction of the free R–module on a set S can be viewed as the left adjoint of the forgetful functor from the category of R–modules to the category of sets. Analogously, there are several forgetful functors from the category of FI–modules, whose left adjoint functors can be viewed as "free" constructions, and which play an important role in the theory.

Definition IV. (The category FB and FB–modules.) Let FB denote the category of finite sets and bijective maps. An FB–*module* over a commutative ring R is a functor from FB to the category of R–modules. A *map of* FB*–modules* is a natural transformation.

Exercise 11. (a) Explain the sense in which an FB–module X is a sequence of S_n–representations X_n, with no additional maps.

(b) Show that a map of FB–modules $F : V \to W$ is a sequence of S_n–equivariant maps $F_n : V_n \to W_n$. What conditions must these maps satisfy?

Exercise 12. (The category of FB–modules.) Fix a commutative ring R. Show that there is a category whose objects are the FB–modules over R and whose morphisms are the FB–module maps.

Definition V. (Induced FI–modules.) Fix a commutative ring R. For fixed $d \in \mathbb{Z}_{\geq 0}$, let W_d be a $R[S_d]$–module. Recall from Exercise 7 that for each n the group S_d also acts on $M(d)_n$ on the right. Define an FI–module $M(W_d)$ by

$$M(W_d)_n = M(d)_n \otimes_{R[S_d]} W_d$$

with an action of the FI morphisms on $M(d)_n$ on the left. More generally, if W is an FB–module (that is, a sequence of S_n–representations), define the FI–module $M(W)$ by

$$M(W) = \bigoplus_{d \geq 0} M(W_d).$$

We call FI–modules of this form *induced* FI*–module*, and $M(W)$ the *induced* FI*–module generated by* W.

Notation VI. (External tensor product of representations.) Let $G \times H$ be a product of groups. Recall that, if U is an G–representation over R and W an H–representation over R, we define the $(G \times H)$–representation $U \boxtimes W$ as follows. As an R–module, $U \boxtimes W \cong U \otimes_R W$, and the group $(G \times H)$ acts by

$$\begin{aligned} (g,h) : U \boxtimes W &\longrightarrow U \boxtimes W \\ u \otimes w &\longmapsto (g \cdot u) \otimes (h \cdot w). \end{aligned}$$

Exercise 13. Fix d and let W_d be an $R[S_d]$–module. Show that, as an S_n–representation,

$$M(W_d)_n \cong \mathrm{Ind}_{S_d \times S_{n-d}}^{S_n} W_d \boxtimes R \qquad \text{with } R \text{ the trivial } S_{n-d}\text{–representation.}$$

Exercise 14. Fix d, and let $R[S_d]$ denote the left regular S_d–representation. Show that there is an isomorphism of FI–modules

$$M(d) \cong M(R[S_d]).$$

Exercise 15. For a FB–module W and a finite set B, show that

$$M(W)_B = \bigoplus_{A \subseteq B} W_A.$$

Exercise 16. Show that the FI morphisms act on $M(W)$ by injective maps.

There is a forgetful functor

$$\mathcal{F} : \mathsf{FI}\text{–Mod} \longrightarrow \mathsf{FB}\text{–Mod}$$

defined by restriction to the subcategory FB $\subseteq$ FI. This forgetful functor takes an FI–module V and remembers only the sequence of $R[S_n]$–modules $\{V_n\}$ and no additional maps. The following exercises show that we may view the assignment $W \mapsto M(W)$ as a functor

$$M(-) : \mathsf{FB}\text{–Mod} \longrightarrow \mathsf{FI}\text{–Mod},$$

and that this functor is a left adjoint to the forgetful functor $\mathcal{F}$.

Exercise 17. ($M(-)$ as a left adjoint.)

(a) Show that the map

$$\begin{aligned} M(-) : \mathsf{FB}\text{–Mod} &\longrightarrow \mathsf{FI}\text{–Mod} \\ W &\longmapsto M(W) \end{aligned}$$

is a covariant functor.

(b) Show that $M(-)$ left adjoint to the forgetful functor $\mathcal{F}$. Concretely, show that for each object $V \in$ FI–Mod and $W \in$ FB–Mod, there is a natural bijection of sets

$$\mathrm{Hom}_{\mathsf{FB}\text{–Mod}}(W, \mathcal{F}(V)) = \mathrm{Hom}_{\mathsf{FI}\text{–Mod}}(M(W), V).$$

(c) Show that the functor $M(-)$ is exact. *Hint:* Exercise 15.

Given this adjunction, we may think of $M(W)$ as the FI–module "freely generated" by the sequence of representations $\{W_n\}$.

Exercise* 18. (FN–modules.) Let FN be the category whose objects are the sets $[n]$, $n \in \mathbb{Z}_{\geq 0}$, and whose only morphisms are the identity morphisms id_n. An FN–set is a functor from FN to the category of sets, that is, it is a sequence of sets A_n. Then there is a forgetful functor

$$\mathsf{FI}\text{–Mod} \longrightarrow \mathsf{FN}\text{–Set}$$

defined by taking an FI–module V to the underlying sequence of sets. Show that this forgetful functor is the right adjoint to the functor

$$\begin{aligned} \mathsf{FN}\text{–Set} &\longrightarrow \mathsf{FI}\text{–Mod} \\ \{A_n\} &\longmapsto \bigoplus_{d \geq 0} M(d)^{\oplus A_d} \end{aligned}$$

Remark VII. Some authors refer to FI–modules of the form $\bigoplus_d M(W_d)$ as *free* FI–modules, and some reserve the term *free* for the more restricted class of FI–modules of the form $\bigoplus_d M(d)^{\oplus c_d}$. In these notes we will not enter into this debate, but refer to these FI–modules as *induced* or *sums of representables*, respectively.

2.2 Perspectives on FI♯-modules

Definition VIII. (Based sets and maps of based sets.) A *based set* A_0 is a set with a distinguished element $0 \in A_0$, called the *basepoint*. A map of based sets $f : A_0 \to B_0$ is a map of sets that takes the basepoint in A_0 to the basepoint in B_0.

Definition IX. (The category FI♯) Let FI♯ (read "FI–sharp") be the category defined as follows. The objects are finite based sets. The morphisms are maps of based sets that are injective away from the basepoints, in the following sense: If $f : A_0 \to B_0$ is map of based sets, then f is an FI♯ morphism if $f^{-1}(b)$ has cardinality $|f^{-1}(b)| \leq 1$ for all $b \in B_0$ not equal to the basepoint.

Notation X. For $n \in \mathbb{Z}_{\geq 0}$, let $[n]_0$ denote the based set

$$[n]_0 := \{0, 1, 2, \dots, n\} \qquad \text{with basepoint } 0.$$

For a finite set A, we write $A_0 := A \sqcup \{0\}$ to be the disjoint union of A with basepoint 0.

Exercise 19.

(a) Show that FI♯ is isomorphic to its opposite category $\text{FI}\sharp^{op}$.

(b) Show that $S_n \subsetneq \text{End}_{\text{FI}\sharp}([n]_0)$, but that S_n is exactly the group of invertible endomorphisms of the object $[n]_0$.

(c) Describe an embedding $\text{FI} \subseteq \text{FI}\sharp$.

(d) Show that the image of every FI morphism in FI♯ has a one-sided inverse.

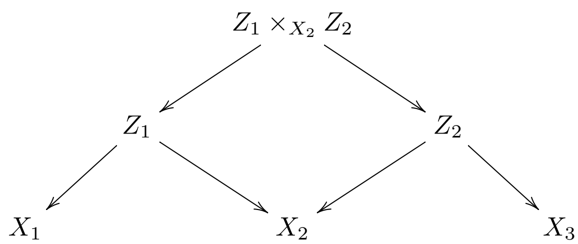
Exercise* 20. (An alternate description of FI♯.) Show that FI♯ is isomorphic to the following category, which was the original description given by Church–Ellenberg–Farb [CEF1, Definition 4.1.1]. The objects are finite sets. The morphisms $\text{Hom}(S, T)$ are triples (A, B, α) with $A \subseteq S$, $B \subseteq T$, and $\alpha : A \to B$ a bijection. The composition of morphisms $(A, B, \alpha) : S \to T$ and $(D, E, \delta) : T \to U$ is the morphism

$$(\alpha^{-1}(B \cap D), \delta(B \cap D), \delta \circ \alpha) : S \to U.$$

Exercise* 21. (FI♯ as the category of spans on FI.) Given a category $\mathcal{C}$, a *span* in $\mathcal{C}$ is a diagram of the form $X_1 \leftarrow Z \to X_2$. Two spans $X_1 \leftarrow Y \to X_2$ and $X_1 \leftarrow Z \to X_2$ are *isomorphic* if there is an isomorphism $Y \cong Z$ in $\mathcal{C}$ making the following diagram commute

$$\begin{array}{ccc} & Y & \\ \swarrow & \downarrow \cong & \searrow \\ X_1 & & X_2 \\ \nwarrow & & \nearrow \\ & Z & \end{array}$$

If the category $\mathcal{C}$ has pullbacks, then we can compose spans $X_1 \leftarrow Z_1 \rightarrow X_2$ and $X_2 \leftarrow Z_2 \rightarrow X_3$ by taking the pullback:



We can then construct a new category, the *category of spans on* $\mathcal{C}$, whose objects are the objects of $\mathcal{C}$, and whose morphisms from X_1 to X_2 are isomorphism classes of spans of the form $X_1 \leftarrow Z \rightarrow X_2$ for some object $Z \in \mathcal{C}$.

(a) Verify that the category of spans on $\mathcal{C}$ is in fact a well-defined category. Identify the identity morphisms, and check that composition of morphisms is associative.

(b) Identify $\mathcal{C}$ as a subcategory.

(c) Show that FI$\sharp$ is equivalent to the category of spans on FI.

Definition XI. (FI$\sharp$–modules.) An FI$\sharp$–module V over a commutative ring R is a functor from FI$\sharp$ to the category of R–modules.

We may write V_n for the value of V on the based set $[n]_0$, or more generally V_A for the value of V on the based set A_0.

An FI^{op}*–module* over a ring R is a functor from the opposite category FI^{op} of FI to the category of R–modules. Equivalently, it is a contravariant functor from FI to R–modules. In the following exercise we will see that an FI$\sharp$–module simultaneously carries an FI– and an FI^{op}–module structure in a compatible way.

Exercise 22. Show that any FI$\sharp$–module is both an FI–module and and FI^{op}–module. Describe what relations must be satisfied by the actions of the FI morphisms and FI^{op} morphisms.

Exercise 23. Let W_d be an $R[S_d]$–module. Show that the FI–module structure on $M(W_d)$ can be promoted to an FI$\sharp$–module structure.

2.3 The functor H_0^{FI}

Definition XII. (The functor H_0^{FI}.) Define a functor

$$H_0^{\mathsf{FI}}(-) : \mathsf{FI}\text{–Mod} \longrightarrow \mathsf{FB}\text{–Mod}$$

$$H_0^{\mathsf{FI}}(V)_n = \frac{V_n}{R[S_n] \cdot \langle \alpha(V_m) \mid \alpha : [m] \to [n] \text{ an FI morphism}, m < n \rangle}$$

In other words, in degree n, the S_n–representation $H_0^{\mathsf{FI}}(V)_n$ is a quotient of V_n, which captures the component of V_n which is not generated in lower FI degree.
When convenient, we will take the codomain of H_0^{FI} to be FI–Mod, and define all non-isomorphism morphisms to act by zero.

Exercise 24. Verify that H_0^{FI} is in fact a functor.

One the key properties of H_0^{FI} is described by the following exercise.

Exercise 25. Let V be an FI–module. Show that V is generated in degree $\leq d$ if and only if

$$H_0^{\mathsf{FI}}(V)_n = 0 \qquad \text{for all } n > d.$$

2.4 A classification of FI♯–modules

The following exercise gives a complete characterization of FI♯–modules. It is a result of Church–Ellenberg–Farb [CEF1, Theorem 4.1.5], and it mirrors an earlier result of Pirashvili [Pir, Theorem 3.1].

Exercise 26. (The structure of the category of FI♯–modules.) The goal of this problem is to show that every FI♯–module has the form $M(W)$ for some FB–module W. Specifically, we will show that an FI♯–module V satisfies a canonical isomorphism $V \cong M(H_0^{\mathsf{FI}}(V))$. *Hint:* Get stuck? Check out Church–Ellenberg–Farb [CEF1, Theorem 4.1.5].

(a) The proof proceeds by induction on n. We will show that given an FI♯–module satisfying

$$V_m = 0 \qquad \text{for all } m < n \tag{$*$}$$

then we can write $V \cong M(V_n) \oplus V'$ for some FI♯–module V' satisfying $V'_m = 0$ for all $m \leq n$. Explain why we then inductively obtain the desired decomposition of our FI♯–module.

(b) For the remainder of the proof, we fix n. Verify that, under Condition $(*)$, $V_n = H_0(V)_n$, so $M(V_n) = M(H_0^{\mathsf{FI}}(V)_n)$.

(c) Let $f : A_0 \to B_0$ be an FI♯ morphism, and suppose that the image of f is m elements plus the basepoint. Show that f factors through the object $[m]_0$.

(d) Given an FI♯–module V satisfying Condition $(*)$, define a map

$$E : V \longrightarrow V$$
$$E_A : V_A \longrightarrow V_A$$
$$E_A = \sum_{\substack{C \subseteq A \\ |C| = n}} (I_C)_*$$

where the morphism $I_C : A_0 \to A_0$ is the identity on the subset $C \subseteq A$, and maps the complement of C to the basepoint. Verify that E is a map of FI♯–modules.

(e) Verify that if $V_n = 0$, then $E : V \to V$ is the zero map.

(f) Given FI♯–modules U and V satisfying Condition $(*)$, and a map of FI♯–modules $F : U \to V$, verify that E commutes with F.

(g) Verify that E is idempotent (that is, $E^2 = E$) in its action on any FI♯–module V satisfying Condition $(*)$. Specifically, show

$$E_A \circ E_A = \sum_{\substack{C, B \subseteq A \\ |C| = |B| = n}} (I_{C \cap B})_* = \sum_{\substack{C \subseteq A \\ |C| = n}} (I_C)_* = E_S.$$

(h) Conclude from part (g) that V decomposes as a direct sum of FI♯–modules $V \cong EV \oplus \ker(E)$, and conclude from part (f) that this decomposition is respected by maps of FI♯–modules satisfying Condition $(*)$.

(i) Verify that $EV_n = V_n$, and $\ker(E)_n = 0$. Our goal is to show that the decomposition $V \cong EV \oplus \ker(E)$ is the desired decomposition $V \cong M(V_n) \oplus V'$.

(j) Let V be an FI♯–module satisfying Condition $(*)$. Construct a map of FI♯–modules $F : M(V_n) \to V$.
Hint: Recall from Exercise 15 that $M(V_n)_B = \bigoplus_{\substack{A \subseteq B \\ |A|=n}} V_A$. Define F_B on the summand V_A to be the map $V_A \to V_B$ induced by the inclusion $A_0 \hookrightarrow B_0$. Verify that this defines a map of FI♯–modules.

(k) Verify that E is the identity map on $M(V_n)$.

(l) Using part (h) and part (k), show that the image $M(V_n)$ is contained in the summand EV.

(m) We therefore have an exact sequence

$$0 \longrightarrow \ker \longrightarrow M(V_n) \xrightarrow{F} EV \longrightarrow \text{coker} \longrightarrow 0.$$

Using part (e) and part (k), show that coker $= \ker = 0$. Conclude that F defines an isomorphism of FI♯–modules from $M(V_n)$ to EV.

(n) Conclude that the decomposition $V \cong EV \oplus \ker(E)$ is precisely a decomposition of the desired form $M(V_n) \oplus V'$ described in part (a).

The following theorem, which appears in Church–Ellenberg–Farb [CEF1, Theorem 4.1.5] is outlined in the exercises.

Theorem XIII. (Classification of FI♯–modules) *Every* FI♯*–module has the form $M(W)$ for some* FB*–module W. In particular, the category of* FI♯*–modules is equivalent to the category of* FB*–modules.*

Exercise 27. (a) Show that, for an FB–module W,

$$H_0^{\mathsf{FI}}(M(W)) = W.$$

(b) Let V be an FI–module. Show that $M\Big(H_0^{\mathsf{FI}}(V)\Big)$ need not equal V. What if V is an FI♯-module?

(c) Show that the functor

$$M(-) : \text{FB–Mod} \longrightarrow \text{FI}\sharp\text{–Mod}$$

is an equivalence of categories, with inverse

$$H_0^{\mathsf{FI}}(-) : \text{FI}\sharp\text{–Mod} \longrightarrow \text{FB–Mod}$$

Exercise 28. Let R be a field of characteristic zero. Conclude from Exercise 27 that the category of FI♯–modules over R is semisimple.

Exercise 29. (Polynomial and exterior algebras as FI♯–modules.)

(a) Let V be the FI–module with $V_n = \mathbb{Z}[x_1, \ldots, x_n]$ and inclusions $V_n \hookrightarrow V_{n+1}$. Show that V is an FI♯–module.

(b) Consider the FI♯–submodules of V consisting of homogeneous degree k polynomials for $k = 0, 1, 2, 3$. Explicity write each of these FI♯–modules in the form $\bigoplus_{d \geq 0} M(W_d)$ for appropriate S_d–representations W_d.

(c) Repeat these exercises for the case that V is the sequence of exterior algebras $V_n = \bigwedge_{\mathbb{Z}} \langle x_1, \ldots, x_n \rangle$.

Next time: We will see that the homology of configuration spaces of points in certain manifolds has an FI♯–module structure. We will exploit this structure to prove that these sequences of homology groups are finitely presented as FI–modules.

Lecture 2: Stability in the homology of configuration spaces

3 Homology of the configuration space of an open manifold as an FI♯–module

3.1 Homology classes in a configuration space

Throughout this lecture, we will fix M to a connected, non-compact manifold of dimension at least 2. The goal of this lecture is to study the homology groups of the configuration spaces $F_n(M)$. Specifically, we will prove the following theorem. This result is originally due (for orientable M) to Church [Chu, Theorem 1] and Church–Ellenberg–Farb [CEF1, Theorem 6.4.3]. The proof presented here is from Miller–Wilson [MW, Theorem 3.12].

Theorem XIV. **($H_q(F_\bullet(M))$ is an FI♯–module generated in degree $\leq 2q$).** *Let M be connected, non-compact smooth manifold of dimension at least 2. Then the FI♯–module $H_q(F_\bullet(M))$ is generated in degree $\leq 2q$.*

These homology groups can be visualized in a very concrete sense. Consider Figure 3.

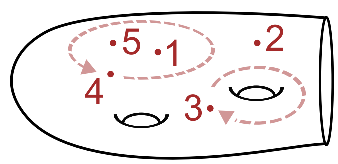


Figure 3: A representative (up to sign) of an element of $H_2(F_5(M))$.

Figure 3 shows two loops in the configuration space of a surface M. Because these two loops do not intersect, they together represent a two-parameter family of points in $F_5(M)$, parameterized by $S^1 \times S^1$. In other words, this figure describes an embedding of a torus into $F_5(M)$. This figure therefore represents (at least up to sign) an element in $H_2(F_5(M))$.

We can view the loop on the right-hand side of Figure 3 as, in a sense, coming from the homology of the underlying manifold M, whereas the loop on the left-hand side as coming from the homology of configurations in $\mathbb{R}^2$. Thus, starting from a knowledge of $H_*(M)$ and $H_*(F_n(\mathbb{R}^d))$, it is possible to generate lots of examples of classes in $H_*(F_n(M)))$.

Understanding the additive relations between these homology classes, however, is nontrivial. In general, the homology groups of the configuration spaces $F_n(M)$ are difficult to compute, and there are few examples of manifolds M where, for example, the Betti numbers are known for all n. Progress has been made recently for the *unordered* configuration spaces of some manifolds; see for instance Knudsen [Knu], Schiessl [Sch], Maguire–Francour [MF], and Drummon-Cole–Knudsen [DCK].

3.2 The homology groups $\{H_*(F_n(M))\}_n$ as an FI♯–module

Even though the homology groups $H_*(F_n(M))$ can be individually difficult to compute, we can gain traction with this problem by bundling these groups $\{H_*(F_n(M))\}_n$ together for all n to form an FI–module (in fact, FI♯–module). This FI–module (or FI♯–module) is sometimes denoted $H_*(F_\bullet(M))$.

We first describe an FI^{op} action on the spaces $F_n(M)$. Since homology is functorial, this structure then defines a co–FI–module structure on the homology groups of $F_n(M)$. The FI^{op} action is illustrated in Figure 4.

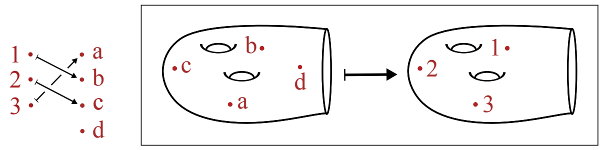


Figure 4: The (contravariant) action of an FI morphism on $\sqcup_n F_n(M)$.

Given an FI morphism f and a configuration in $F_n(M)$, points in the image of the f are relabelled by their preimage, and points not in the image of f are forgotten.

Exercise 30. Verify that this FI^{op} action on $\sqcup_n F_n(M)$ is functorial.

To define a covariant action of FI, we will use the assumption that M is non-compact. It turns out that this implies the existence of an embedding $e : M \sqcup \mathbb{R}^{\dim(M)} \hookrightarrow M$ such that $e|_M$ is isotopic to the identity. See Figure 5.

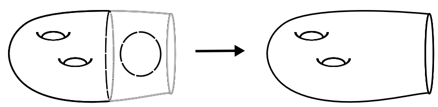


Figure 5: An embedding $e : M \sqcup \mathbb{R}^{\dim(M)} \hookrightarrow M$.

To define the FI action, we fix such an embedding e. (The action does depend on the choice of e, but any choice will do.) Unlike the FI^{op} action, the FI action is only defined up to homotopy. Since homotopic maps define the same map on homology, however, we obtain a well-defined FI–module structure on $H_*(F_n(M))$.

The FI action is illustrated in Figure 6.

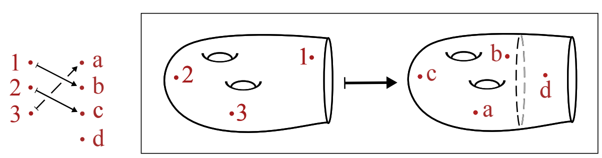


Figure 6: The (covariant) action up to homotopy of an FI morphism on $\sqcup_n F_n(M)$.

Given an FI morphism f and a configuration in $F_n(M)$, the configuration is mapped to its image under the embedding $e|_M$, and points are relabelled by their image under f. For each element in the codomain of f that is not in its image, a labelled point is introduced in $e(\mathbb{R}^{\dim(M)})$.

Exercise 31. Verify that this FI action on $\sqcup_n F_n(M)$ is functorial (up to homotopy).

The FI–module and co–FI–module structures on $H_*(F_n(M))$ are compatible, and extend to an action of FI♯ on the homology groups, as established in the following exercises.

Exercise 32. Verify that the FI– and co–FI–module structures on $H_*(F_\bullet(M))$ extend to an FI♯–module structure. Describe the action of a general FI♯ morphism.

4 Representation stability for the homology of (ordered) configuration spaces

4.1 Historical results: stability in the homology of (unordered) configuration spaces

Our objective is to prove representation stability for the homology groups $\{H_*(F_n(M))\}_n$. These results were inspired by classical stability results for the *unordered* configuration spaces, due to McDuff [McD] and Segal [Se].

Just as we defined a map $F_n(M) \to F_{n+1}(M)$ by introducing a $(n+1)^{st}$ point "at infinity", it is possible to define a continuous map $C_n(M) \to C_{n+1}(M)$ by introducing an unlabelled point "at infinity", as in Figure 7.

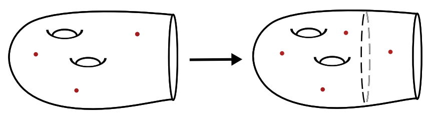


Figure 7: The stabilization map $t : C_n(M) \to C_{n+1}(M)$.

McDuff proved the folloiwng result stability result, and Segal determined the stable range.

Theorem XV. (Classical homological stability for unordered configuration spaces). *Let M be a connected, non-compact manifold of dimension at least 2. Then the stabilization map $t : C_n(M) \to C_{n+1}(M)$ induces isomorphisms on homology*

$$t_* : H_q(C_n(M)) \xrightarrow{\cong} H_q(C_{n+1}(M)) \qquad \text{for all } n \geq 2q.$$

Exercise 33. Fix M.

(a) Recall that there is a covering map $F_n(M) \to C_n(M)$. Explain why there is an isomorphism between $H_q(C_n(M); \mathbb{Q})$ and the S_n–coinvariants $H_q(F_n(M); \mathbb{Q})_{S_n}$.
Hint: See the *transfer map* in (for example) Hatcher [H2, Section 3.G].

(b) Show that the rational homological result implied by Theorem XV implies that the dimension of the isotypic component of the trivial representation in $H_q(F_n(M); \mathbb{Q})$ stabilizes for $n \geq 2q$. Thus (at least when working rationally), we can view Theorem XIV as a generalization of Theorem XV.

4.2 The spectral sequence

We now turn our attention to proving the main result: Theorem XIV, representation stability for the homology of (ordered) configuration spaces. To do this, we will use (for each n) a spectral sequence called the *arc resolution spectral sequence*.

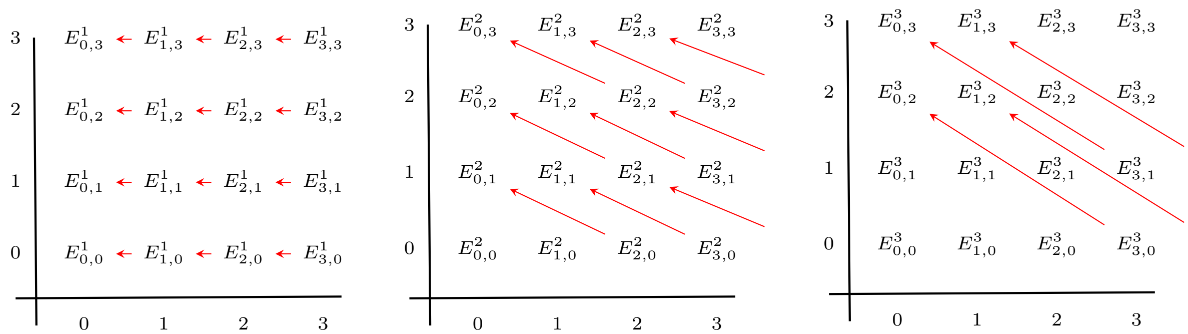
A review of homology spectral sequences

Recall that a (homology) spectral sequence is a sequence of bigraded abelian groups $E^r = \bigoplus_{p,q} E^r_{p,q}$, called *pages*, for $r = 0, 1, 2, \ldots$. Each page has a differential map $d^r : E_r \to E_r$ satisfying $(d^r)^2 = 0$, and the page E^{r+1} is the homology of the complex (E^r, d^r), in the sense that

$$E^{r+1}_{p,q} = \frac{\text{kernel of } d^r \text{ at } E^r_{p,q}}{\text{image of } d^r \text{ in } E^r_{p,q}}.$$

In particular $E^{r+1}_{p,q}$ is always a subquotient of $E^r_{p,q}$. For our spectral sequence, the differentials satisfy

$$d^r : E^r_{p,q} \longrightarrow E^r_{p-r,q+r-1}.$$



The pages E^1, E^2, and E^3.

Our spectral sequence has the property that the groups $E^r_{p,q}$ can be nonzero only when $p \geq -1$ and $q \geq 0$. This implies that, at any fixed point (p, q), for r sufficiently large, either the domain or the codomain of any differential d^r to or from $E^r_{p,q}$ will be zero. Hence, for r large we find (upon taking homology)

$$E^r_{p,q} = E^{r+1}_{p,q} = E^{r+2}_{p,q} = \cdots$$

Recall that, in general, we call this stable group $E^\infty_{p,q}$, and call the bigraded abelian group $E^\infty_{*,*}$ the *limit* of the spectral sequence. In general the sequence of groups $\{E^r_{p,q}\}_r$ converges at a page r that depends on (p, q). If there is some r such that $E^r_{p,q} = E^\infty_{p,q}$ for all p and q, then we say that the spectral sequence *collapses* on page E_r.

The arc resolution spectral sequence

The following spectral sequence is described in Miller–Wilson [MW, Proposition 3.8].

Proposition XVI. (The E^2 page of the arc resolution spectral sequence). *Let M be a noncompact connected smooth manifold of dimension at least two. Fix n, and fix a set S of size n. The arc resolution spectral sequence satisfies:*

$$E^2_{p,q}(S) \cong \bigoplus_{\substack{S=P\sqcup Q,\\ |P|=p+1}} \mathcal{T}_P \otimes H_0^{\mathsf{FI}}(H_q(F_\bullet(M)))_Q \qquad \textit{for } p \geq -1 \textit{ and } q \geq 0$$

$$\cong \mathit{Ind}^{S_n}_{S_{p+1}\times S_{n-p-1}} \mathcal{T}_{p+1} \boxtimes H_0^{\mathsf{FI}}(H_q(F_\bullet(M)))_{n-p-1}.$$

for certain combinatorially-defined groups $\mathcal{T}_{p+1}$. (The precise definition of these groups is not needed for our proof, but we note that $\mathcal{T}_1 = 0$.)

In particular, the leftmost E^2 column $p = -1$ are the FI–homology groups

$$E^2_{-1,q}(n) \cong H_0^{\mathsf{FI}}(H_q(F_\bullet(M)))_n.$$

The page $E^2_{p,q} = 0$ for $p < -1$ or $q < 0$.

4	$H_0^{\mathsf{FI}}\Big(H_4(F_\bullet(M))\Big)_6$	0	$\mathrm{Ind}_{S_2\times S_4}^{S_6}\mathcal{T}_2 \boxtimes H_0^{\mathsf{FI}}(H_4(F_\bullet(M)))_4$	$\mathrm{Ind}_{S_3\times S_3}^{S_6}\mathcal{T}_3 \boxtimes H_0^{\mathsf{FI}}(H_4(F_\bullet(M)))_3$
3	$H_0^{\mathsf{FI}}\Big(H_3(F_\bullet(M))\Big)_6$	0	$\mathrm{Ind}_{S_2\times S_4}^{S_6}\mathcal{T}_2 \boxtimes H_0^{\mathsf{FI}}(H_3(F_\bullet(M)))_4$	$\mathrm{Ind}_{S_3\times S_3}^{S_6}\mathcal{T}_3 \boxtimes H_0^{\mathsf{FI}}(H_3(F_\bullet(M)))_3$
2	$H_0^{\mathsf{FI}}\Big(H_2(F_\bullet(M))\Big)_6$	0	$\mathrm{Ind}_{S_2\times S_4}^{S_6}\mathcal{T}_2 \boxtimes H_0^{\mathsf{FI}}(H_2(F_\bullet(M)))_4$	$\mathrm{Ind}_{S_3\times S_3}^{S_6}\mathcal{T}_3 \boxtimes H_0^{\mathsf{FI}}(H_2(F_\bullet(M)))_3$
1	$H_0^{\mathsf{FI}}\Big(H_1(F_\bullet(M))\Big)_6$	0	$\mathrm{Ind}_{S_2\times S_4}^{S_6}\mathcal{T}_2 \boxtimes H_0^{\mathsf{FI}}(H_1(F_\bullet(M)))_4$	$\mathrm{Ind}_{S_3\times S_3}^{S_6}\mathcal{T}_3 \boxtimes H_0^{\mathsf{FI}}(H_1(F_\bullet(M)))_3$
0	$H_0^{\mathsf{FI}}\Big(H_0(F_\bullet(M))\Big)_6$	0	$\mathrm{Ind}_{S_2\times S_4}^{S_6}\mathcal{T}_2 \boxtimes H_0^{\mathsf{FI}}(H_0(F_\bullet(M)))_4$	$\mathrm{Ind}_{S_3\times S_3}^{S_6}\mathcal{T}_3 \boxtimes H_0^{\mathsf{FI}}(H_0(F_\bullet(M)))_3$
	-1	0	1	2

Figure 8: $E^2_{p,q}(6) \cong \mathrm{Ind}_{S_{p+1}\times S_{6-p-1}}^{S_6}\mathcal{T}_{p+1} \boxtimes H_0^{\mathsf{FI}}(H_q(F_\bullet(M)))_{6-p-1}$.

We note that this description of the E^2 page uses the $\mathsf{FI}\sharp$–module structure on the homology groups.

Exercise 34. Show that $H_0(F_\bullet(M)) \cong M(0)$ as an FI–module. Deduce that the bottom $q = 0$ row of the E^2 page vanishes except when $p = n - 1$.

Our goal is to show $H_q(F_\bullet(M))$ is generated as an FI–module in degree $\leq 2q$. Thus by Exercise 25, it suffices to show that the first $p = -1$ column of the arc resolution spectral sequence vanishes at $E_{-1,q}$ for all $n > 2q$.

The key is the following result, which follows from the Appendix of Kupers–Miller [KM]: the arc resolution converges to zero in a range, with $E^\infty_{p,q}(n)$ vanishing for all n large enough relative to (p, q).

Proposition XVII. (The E^∞ page of the arc resolution spectral sequence). *Let M be a noncompact connected smooth manifold of dimension at least two. Fix n. Then*

$$E^\infty_{p,q}(n) = 0 \qquad \text{for all } (p,q) \text{ with } p + q \leq n - 2.$$

4.3 The proof

We now have all the necessary ingredients to prove the main theorem, Theorem XIV.

Exercise 35. (a) Show that $H_0^{\mathsf{FI}}\Big(H_0(F_\bullet(M))\Big)_n = 0$ for all $n > 0$.

(b) Use the arc resolution spectral sequence to proceed by induction on q, to show that

$$H_0^{\mathsf{FI}}\Big(H_q(F_\bullet(M))\Big)_n = 0 \qquad \text{for all } n > 2q.$$

Hint: Assuming by induction that the statement holds in homological degree $i < q$, what are the possible differentials to or from $E^r_{-1,q}(n)$, for $n > 2q$? What is $E^\infty_{-1,q}(n)$? What can you conclude about

$$E^2_{-1,q}(n) \cong H_0^{\mathsf{FI}}(H_q(F_\bullet(M)))_n?$$

Conclude Theorem XIV: $H_q(F_\bullet(M)$ is an $\mathsf{FI}\sharp$–module generated in degree $\leq 2q$.

Exercise* 36. Read Miller–Wilson [MW, Sections 2.2, 3.2]. Explain how to construct the arc resolution spectral sequence as the spectral sequence associated to the semi-simplicial space, the *arc resolution*.

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Comments / corrections welcome!

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