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Introduction to Buildings I: Coxeter groups & reflection groups

Intro

Building - geometric object, introduced by Tits
goal: understand semisimple complex Lie gps.

Now: play a role in study of
- alg. gps over arbitrary fields
- geometric gp theory
- differential geometry ...

Perspectives - based on concepts of chamber.

our focus
- simplicial (original, Tits)
  - bldg is a simplicial complex
  - chamber - top dimensional cell
- combinatorial (modern, Tits)
  - bldg is abstract combinatorial object
  - chamber - element of abstract set encoded by chamber system (adjacency graph)
- metric (Davis)
  - bldg is CAT(0) metric space
  - chamber - metric space.

A bldg is a union of subcomplexes called apartments
- Bldg of spherical type - apartments are top. spheres
- Bldg of affine / Euclidean type - apartments are subdivisions of affine space.

Spherical Bldg: Prototypical example

Flags in $\mathbb{C}^n$

vertices $\longleftrightarrow$ proper nonzero subspaces of $\mathbb{C}^n$

edges $\longleftrightarrow$ inclusion $V_0 \subsetneq V_1$

chambers $\longleftrightarrow$ complete flags $0 \subsetneq V_0 \subsetneq \cdots \subsetneq V_{n-2} \subsetneq \mathbb{C}^n$

((n-2)-simplices)

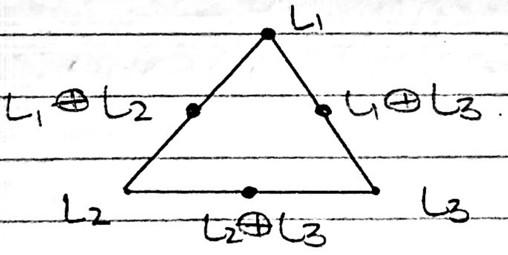
apartment ((n-2)-spheres)

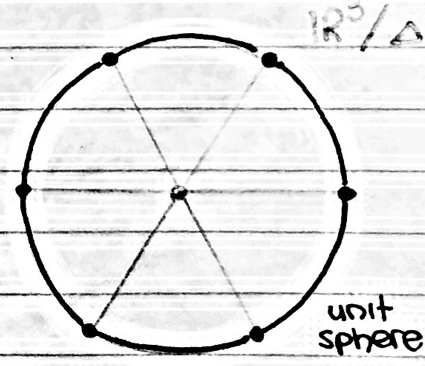
- indexed by decompositions $L_1 \oplus L_2 \oplus \cdots \oplus L_n = \mathbb{C}^n$ into lines
- apartment is full subcomplex on subspaces spanned by all nonempty proper subsets of $\{L_1, \ldots, L_n\}$.

Eg $n=3$ (type A_2)

apartment for

$L_1 \oplus L_2 \oplus L_3 = \mathbb{C}^3$.





Root system for S_3

$\underline{\text{Defn}}$ A <u>Coxeter Gp</u> G is a gp with a presentation of the following form:

$$G = \langle s \in S \mid s^2, (st)^{m_{s,t}} \rangle$$

generators are involutions

For $s \neq t$
$m_{s,t} \in \mathbb{Z}_{\geq 2} \cup \{\infty\}$

$m_{s,t} = 2 \Leftrightarrow$ s and t commute

$m_{ss} = 1$.

The pair (G, S) is called a <u>Coxeter system.</u>

<u>Motivating Examples:</u>

① The Symmetric Gp S_n (Type A_{n-1})

$$S_n = \langle s_1, \ldots, s_{n-1} \mid s_i^2, \; (s_is_j)^2 \text{ if } |i-j|>1, \; (s_is_j)^3 \text{ if } |i-j|=1 \rangle$$

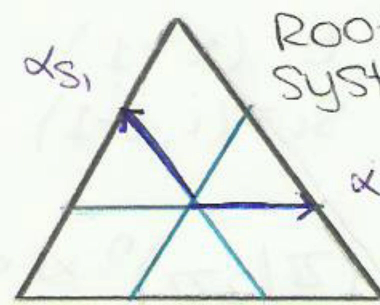
$s_i = (i\ i{+}1)$

$(s_is_j)^3$ called the <u>braid relations.</u>

Rank: $n-1$

Degrees: $2, 3, 4, \ldots, n$

<u>Eg.</u> $S_3 = \langle s_1, s_2 \mid s_1^2, s_2^2, (s_1s_2)^3 \rangle$.



$S_3 = Di_3$

s_1 = reflection along α_{s_1}
(ie, in hyperplane perp to α_{s_1})
s_2 = reflection along α_{s_2}
s_1s_2 = rotation by $2\pi/3$.

② Dihedral Gp Di_m (acting on m-gon).

$$Di_m = \langle s, t \mid s^2, t^2, (st)^m \rangle$$

Rank 2

Degrees: 2, m.

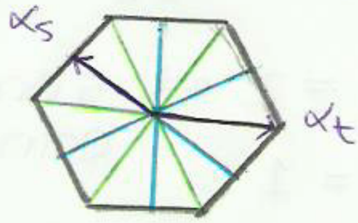
s = reflection along α_s

t = reflection along α_t

st = rotation by $2\pi/m$.

Eg.

$$Di_5 = \langle s, t \mid s^2, t^2, (st)^5 \rangle$$



$$Di_6 = \langle s, t \mid s^2, t^2, (st)^6 \rangle$$

$Di_6 \cong G_2$

Note that the nature of the root system differs in odd and even degree m.

For m even, there are roots of two different lengths.

③ Hyperoctahedral Gp (Gp of signed permutation matrices).

Type B_n | C_n.

$$B_n = \langle t, s_1, \ldots, s_{n-1} \mid s_i^2, t^2, (s_1 t)^4, (s_i t)^2 \text{ for } i > 1, \; (s_i s_j)^3 \text{ if } |i-j| = 1, \; (s_i s_j)^2 \text{ if } |i-j| < 1 \rangle$$

$t = (1\ {-1})$

$s_i = (i\ \ i{+}1)$

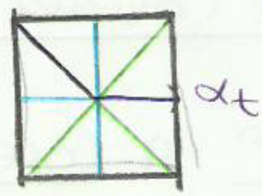
Rank n

$B_n = (\mathbb{Z}/2\mathbb{Z})^n \rtimes S_n$.

Degrees $2, 4, 6, \ldots, n$.

B_n is the symmetry gp of an n-cube.

Eg $B_2 = \langle t, s \mid s^2, t^2, (st)^4 \rangle$.



$B_2 \cong Di_4$

The Theory of Coxeter Gps

Defn $|S| = n$ is the rank of a Coxeter gp G

Given a Coxeter system (G, S), there is a procedure for associating a root system to (G, S), which gives G the structure of reflection gp on $\mathbb{R}^n$, $n = |S|$.

- Take the $\mathbb{R}$-vector space with basis $\{\alpha_s \mid s \in S\}$. These are called the simple roots.
- Define action of G so that $s \in S$ acts by
$$\alpha_t \cdot s = \alpha_t + 2\cos\left(\frac{\pi}{m_{s,t}}\right)\alpha_s.$$

s acts by a "reflection along α_s" in the sense that α_s is an eigenvector of s with eigenvalue -1, and has a direct complement on which s acts trivially.

(Check: Its trace is $n-2$, and its minimal polynomial is determined by $s^2 = \text{identity}$)

Defn The orbit $\Phi = \{\alpha_s \cdot G\}$ is the root system associated to (G, S). Its elements are roots.

For G finite, can define inner product on V making reflections orthogonal

Reflection Groups

V - fin-dim inner product space, $V \cong \mathbb{R}^n$

$G \curvearrowright V$ $\quad$ G discrete

G is a reflection gp if H is generated by a set S of reflections

Notation: H hyperplane

S_H = reflection in H, ie, s acts on H by 1, s acts on $H^\perp$ by -1

Fact: Coxeter gps need not be finite but

$\{$finite Coxeter gps$\}$ = $\{$finite reflection gps$\}$

Exercise: $G \curvearrowright V$ reflection gp, G finite

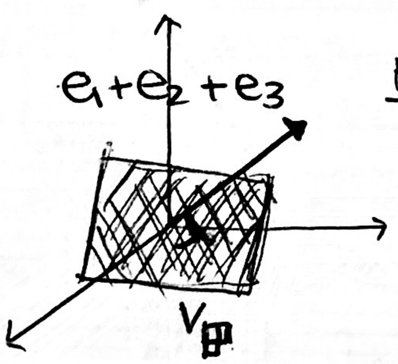
$G = \langle s_H \mid H \in \mathcal{H} \rangle$

Then

- fixed set $V^G = \bigcap_{H \in \mathcal{H}} H$
- $V = V^G \oplus \underbrace{(V^G)^\perp}_{V_1}$, $\quad$ • V' G-invariant
- $G \curvearrowright V'$ as reflection gp
- $(V')^G = 0$

V' is the essential part of V.

The action of G is essential if $V^G = 0$



Eg $S_3 \curvearrowright \mathbb{R}^3 = \langle e_1, e_2, e_3 \rangle$ canonical rep.

$(\mathbb{R}^3)^{S_3} = \text{span}_{\mathbb{R}}(e_1+e_2+e_3)$

perp: $\{a_1e_1 + a_2e_2 + a_3e_3 \mid a_1+a_2+a_3 = 0\}$

$\uparrow$

essential part.

<u>Exercise</u> $G \curvearrowright V$, $G' \curvearrowright V'$ finite reflection gps

Then $\exists$ action $(G \times G') \curvearrowright V \oplus V'$ as reflection gp.

G is <u>irreducible</u> if action cannot decompose as product.

Cell Decomposition

$G \curvearrowright V$ essential finite reflection gp

$\mathcal{H} = \{H_1, \ldots, H_k\}$ – set of hyperplanes H st $S_H \in G$.

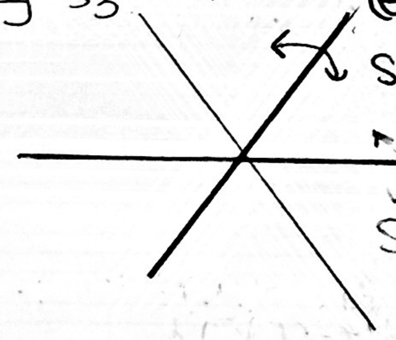
<u>Exercise</u> $\mathcal{H}$ = G-orbit of hyperplanes fixed by generators

Define f_i – nonzero linear functional $\bar{w}$ kernel H_i.

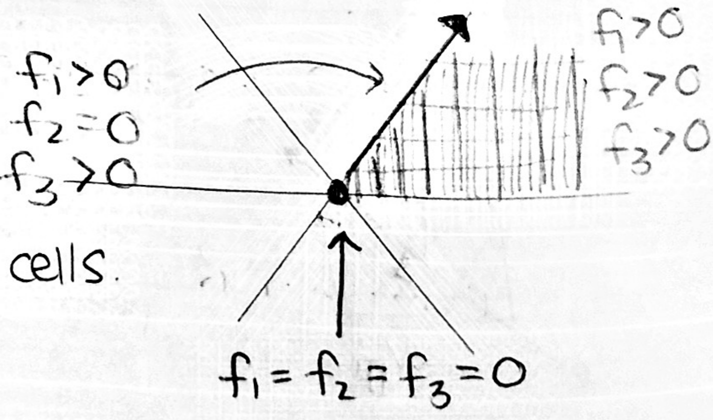
<u>Def^n</u> <u>cell</u> in V – nonempty set $A \subseteq V$ defined by choosing for each i one of the conditions

$$f_i = 0, \quad f_i > 0, \quad \text{or} \quad f_i < 0.$$

<u>Eg</u> $S_3 \curvearrowright V$



<u>NB</u> Many sign choices are inconsistent and yield empty set



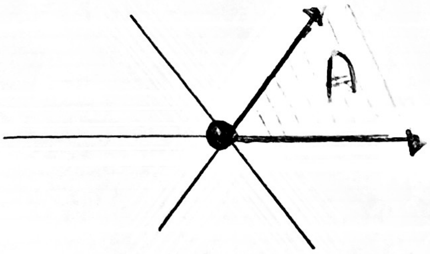
13 cells.

Def^n Cell A

Support of A – linear subspace L_A defined by equalities $f_i = 0$ in conditions defining A

Exercise A is open in L.
A is called an open cell

Def^n Face of A – cell obtained by replacing 0 or more inequalities with equalities



A has 4 faces (including A)

Def^n closed cell $\overline{A}$ – obtained from cell A by replacing strict inequalities with weak.

Exercise intersection of closed cells is a closed cell.

Def^n chamber – cell of dim n
- defined by $f_i > 0$ or $f_i < 0$ $\forall i$
- support is V

Given chamber C

panel of C – codim 1 face
wall of C – support of panel (hyperplane in $\mathcal{H}$)

Fact $G \curvearrowright V$ essential finite reflection gp, $V \cong \mathbb{R}^n$
- Chamber C

- chambers C are simplicial cones,
  ie, $C = \{ \sum \lambda_i e_i \mid \lambda_i > 0 \}$
  for some basis $e_1, \dots, e_n$ of V
- chambers have n panels
- The functionals $f_1, \dots, f_n$ are a basis for V^*
- G acts simply transitively on set of chambers,
- $|G| = \#$chambers
- C any chamber, $G = \langle s_H \mid H \text{ wall of } C \rangle$

Posets and simplicial complexes.

Simplicial complex $X \rightsquigarrow$ poset of simplices of X and $\emptyset$ under inclusion.

Brown says a poset P "is" a simplicial complex if it arises from one in this way.

Sufficient conditions:

(1) $\forall\ A, B \in P$, A, B have greatest lower bound.

(2) $\forall\ A \in P$, $P_{\leq A}$ is isomorphic to the poset of subsets of $\{1, 2, \dots, r\}$ for some r.
r is called the <u>rank</u> of A.

Reconstruct complex: vertices $\leftrightarrow$ rank-1 elements
Glue simplex A to the set of vertices in $P_{\leq A}$.

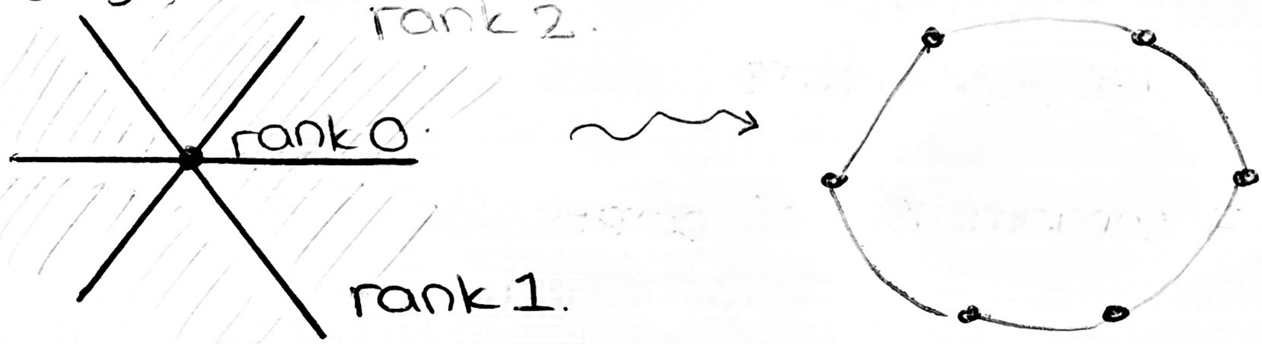
<u>Warning</u>: Brown calls this the geometric realization
This is <u>not</u> the usual meaning of geometric realization for a poset.

$G \curvearrowright V$

Thm G fin. reflection gp, essential
Σ = poset of cells of V under inclusion.
Then Σ "is" a simplicial complex.

Pf Condition (1) - $\overline{A} \cap \overline{B}$ is a closed cell
Condition (2) - suffices to check for chambers; they are simplicial cones.

Eg S_3.



Thm $\Sigma \cong S^{n-1}$

Geometric description of Σ

Thm Σ is a triangulation of unit sphere $S^{n-1} \subseteq V$

realized by radial projection of cells

$$\left.\begin{array}{ccc} V \setminus \{0\} & \longrightarrow & S^{n-1} \\ A & \longmapsto & A \cap S^{n-1} \end{array}\right\} \text{homeo of simplicial complexes}$$

Group-theoretic description of Σ

S - reflections generating G.

Defn standard parabolic subgp of G
- gp generated by subgp $S' \subseteq S$

Thm $G \curvearrowright V$ essential finite reflection gp.
C - chamber
S - reflections in walls of C

- $\overline{C}$ = set of reps for G-orbits in V
- For $x \in \overline{C}$,
  stabilizer $G_x = \langle S_x \rangle$, $S_x = \{ s \in S \mid sx = x \}$
  ↑ standard parabolic subgp.
- A - open cell containing x
  then G_x fixes A pointwise.
- A cell, G_A setwise stabilizer
  G_A fixes A pointwise.

Thm $\exists$ isomorphism of posets compatible with G-action.

$$\Sigma_{\leq C} \xrightarrow{\cong} \left(\begin{matrix} \text{standard parabolic} \\ \text{subgps} \end{matrix} \right)^{op}$$
$$A \longmapsto G_A$$

$$\Sigma \xrightarrow[\cong]{} \left(\begin{matrix} \text{cosets of standard} \\ \text{parabolic subgps} \end{matrix} \right)^{op}$$

Map $A \longmapsto G_A$ has inverse
$G' \longmapsto$ fixed-point-set of G' in C.