Introduction to Buildings II: The definition of a building

Jenny Wilson
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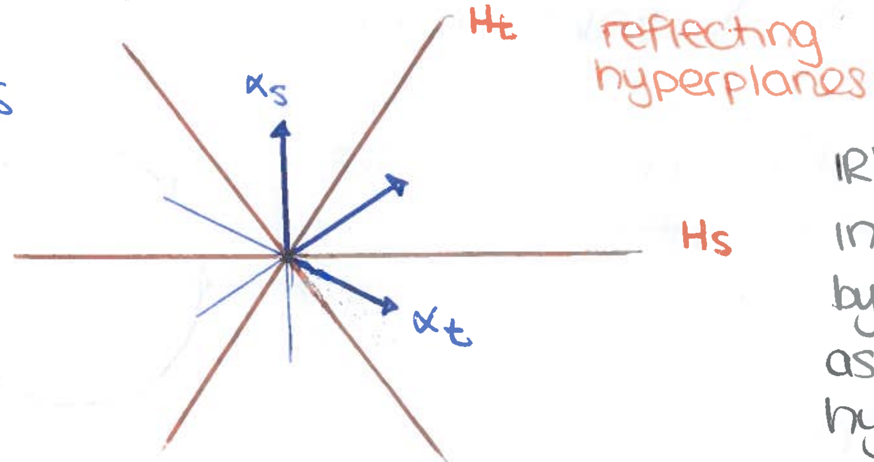
Review of last time

Coxeter gp $G = \langle s \in S \mid s^2, (st)^{m_{s,t}} \rangle$ — encoded by $[m_{s,t}]$ = Coxeter matrix

$|S| = n < \infty$ rank of G

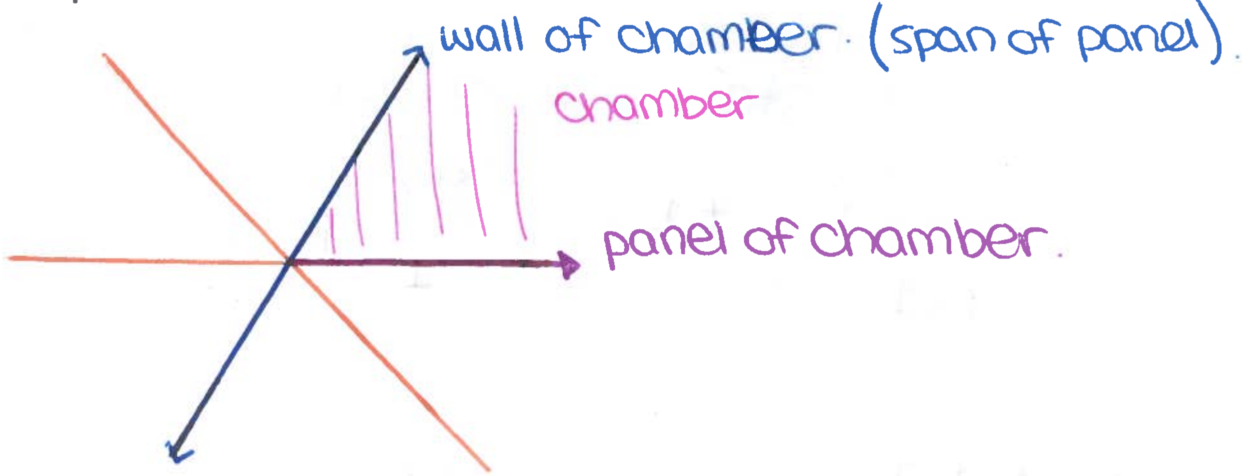
Construct action $G \curvearrowright \mathbb{R}^n$, $s \in S$ act by reflections.
(For $|G| < \infty$, can define inner product so s acts by orthogonal reflection)



G finite: $\mathbb{R}^n$ decomposes into cones defined by linear functionals associated to each hyperplane.

$S_3 = \langle s,t \mid s^2, t^2, (st)^3 \rangle$

For $|G|$ finite:

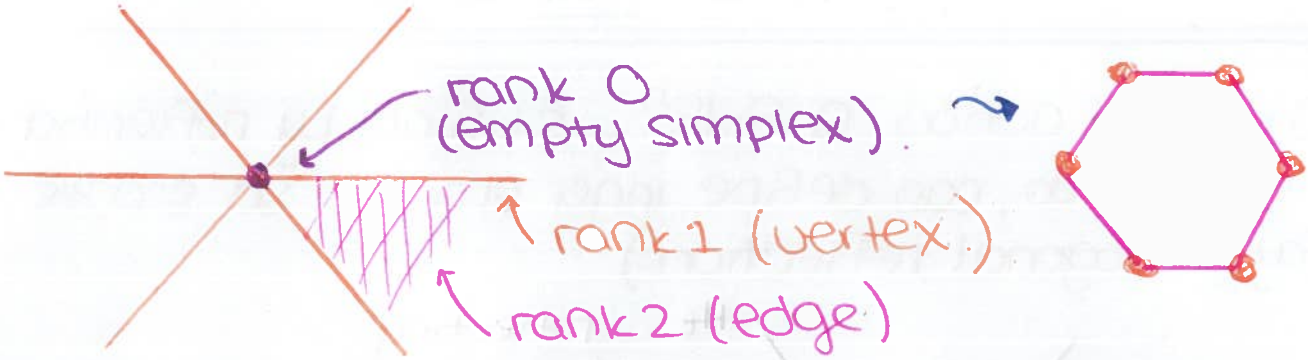


- G acts simply transitively on set of chambers
  so given distinguished choice of chamber ("fundamental chamber") get bijection $G \longleftrightarrow \{\text{chambers}\}$.
- G is generated by reflections in walls of fund. chamber

Coxeter complexes (G finite)

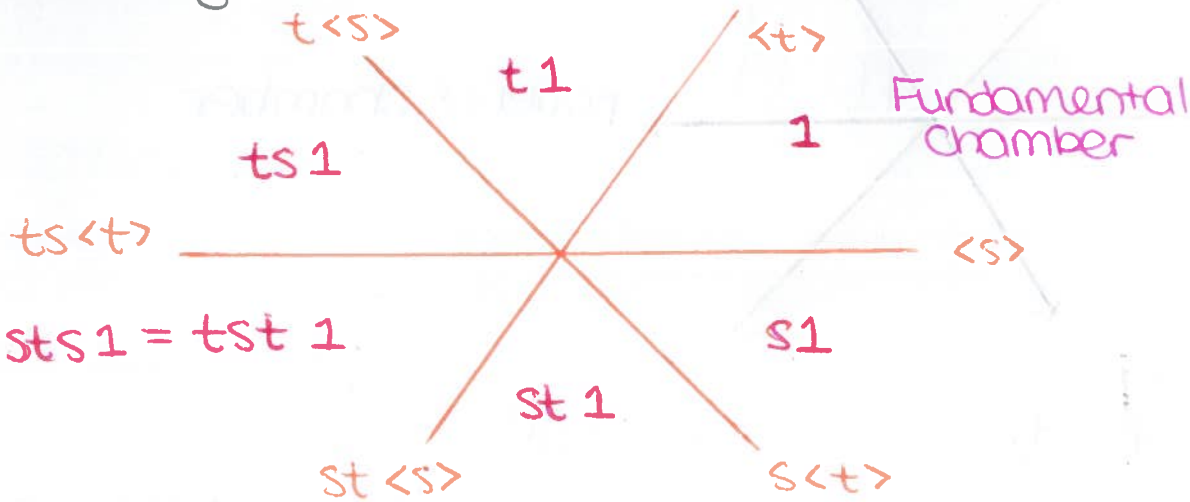
3 characterizations:

1. "geometric realization" (in sense of Brown) of poset of cells under inclusion:



2. simplicial structure on unit sphere $S^{n-1} \subseteq \mathbb{R}^n$ induced by intersection with cells

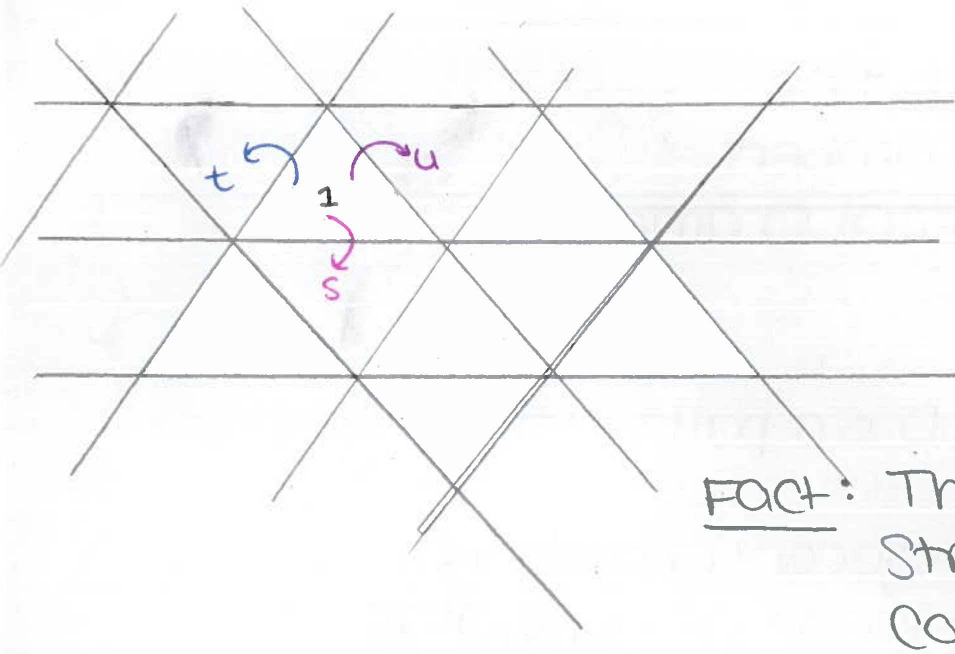
3. "geometric realization" of poset of cosets of standard parabolic subgps of G under reverse inclusion (identify with stabilizers of cells)



Definition 3 of the complex also makes sense for infinite G!

Some Coxeter complexes of infinite gps G:

Ex $G = \langle s, t, u \mid s^2, t^2, u^2, (st)^3, (tu)^3, (su)^3 \rangle$



G = affine reflection gp associated to tiling of plane by equilateral triangles

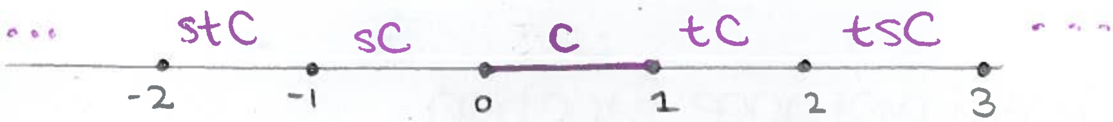
Fact: This tiling (as simplicial structure) is the Coxeter complex

Ex $G = \langle s, t \mid s^2, t^2 \rangle$ infinite dihedral gp

$G \curvearrowright \mathbb{R}$

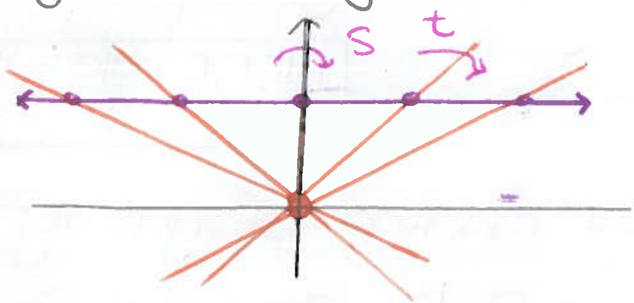
$s \mapsto [x \mapsto -x]$ reflection about 0

$t \mapsto [x \mapsto 2-x]$ reflection about 1



This is the Coxeter complex

Can linearize action by embedding $\mathbb{R}$ in $\mathbb{R}^2$:



NB: This is dual to the action of G assoc to root system $s = \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix}$, $t = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}$

Both s,t fix hyperplane span $(1,1)$.

Metrics and W-Metrics

<u>Defn</u> Chambers C, C' are <u>adjacent</u> if they have a common panel.
$\Leftrightarrow$ C is mapped to C' by generator of G

The <u>chamber graph</u> has:
- vertices – chambers
- edges – adjacent chambers

C, D – chambers.
A <u>gallery</u> from C to D is a path in the chamber graph
(ie, a sequence of adjacent chambers
equivalently, a sequence of generators of G)

The <u>gallery distance</u>
$d(C,D)$ = length of minimal gallery from C to D

<u>Fact</u> $d(C,D) = \#$walls separating C from D.

The <u>Weyl distance function</u>
$$\delta : \{\text{chambers}\} \times \{\text{chambers}\} \longrightarrow G$$

For C, D, let $(s_1, s_2, \dots, s_k)$ be sequence of reflections associated to minimal gallery.

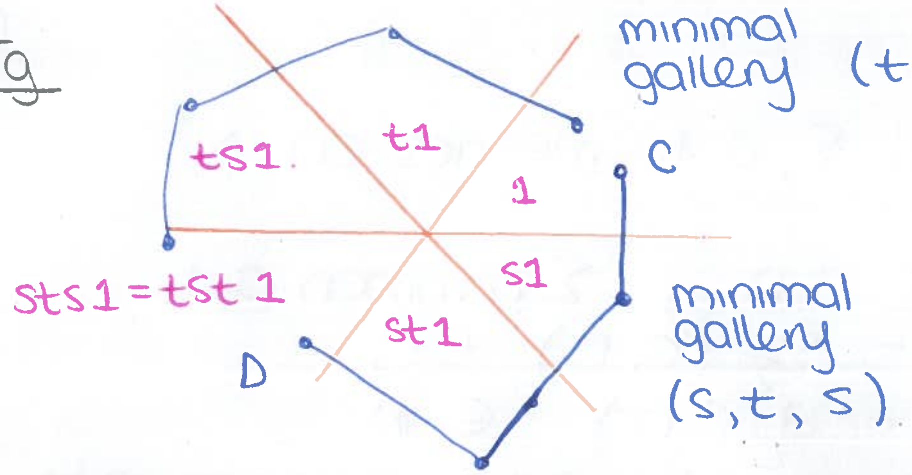
Then $\delta(C,D) = s_1 s_2 \cdots s_d \in G$

<u>NB</u> C, D adjacent $\Leftrightarrow \delta(C,D) \in S$

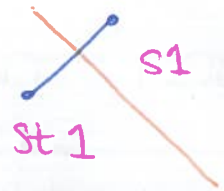
<u>Fact</u>: This is independent of choice of gallery.

<u>Fact</u> $\delta(C,D) = w$, $\left\{ \begin{array}{c} \text{reduced words} \\ \text{for } w \end{array} \right\} \xleftrightarrow{\text{bij}} \left\{ \begin{array}{c} \text{minimal} \\ \text{galleries} \\ C \to D \end{array} \right\}$.

Eg



Convention:



These two chambers differ by right mult by t, so this edge corresponds to t.

Then $d(C,D)$ = word length of $\delta(C,D)$ wrt generators S

Buildings

A *building* is a simplicial complex Δ which can be expressed as a union of subcomplexes Σ (apartments) st

- (B0) Each Σ is a Coxeter complex
- (B1) $\forall$ pairs of simplices $A, B \in \Delta$, $\exists$ apartment containing both
- (B2) Given two apts Σ, Σ' containing two simplices A, B (possibly $A, B = \emptyset$), $\exists$ iso $\Sigma \longrightarrow \Sigma'$ fixing A & B pointwise.

A *system of apartments* is a collection of subcomplexes Σ satisfying the axioms.

Fact: Can recover Coxeter gp G from Coxeter complex
generators $\leftrightarrow$ walls (dually, vertices) of fundamental chamber

Key: $\forall s,t \in S$, $\langle s,t \rangle$ is dihedral gp $D_{m(s,t)}$.
Can identify its Coxeter complex as subcomplex ("link of simplex of cotype s,t")
$m_{s,t}$ is its diameter (in sense of metric d)

Fact: Gp G is independent of choice of system of apts.

<u>Fact</u> Can extend δ, d to metrics on Δ

Given chambers C, D, Σ common apt

$$\delta(C,D) = \delta_\Sigma(C,D) \in G$$
$$d(C,D) = d_\Sigma(C,D) \in \mathbb{N}$$

are well-defined, $d(C,D)$ = gallery distance in Δ.

<u>Fact</u> The combinatorial approach to buildings:

Given Coxeter gp (W,S), data of bldg is

- a nonempty set $\mathcal{C}$ (the chambers)
- a weyl distance function $\delta : \mathcal{C} \times \mathcal{C} \to W$

subject to some axioms.

<u>Example of a building</u>

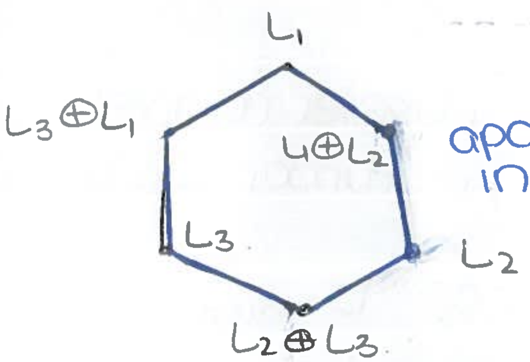
Flag complex Δ_n: V - n-dim^l vector space

vertices $\longleftrightarrow$ nonempty proper subspaces

simplices $\longleftrightarrow$ flags

apartments $\longleftrightarrow$ given frame $\underbrace{L_1 \oplus L_2 \oplus \cdots \oplus L_n}_{\text{lines}} = V$,

Σ = full subcomplex on spans of proper nonempty subsets of $\{L_1, L_2, \ldots, L_n\}$



apartment in Δ_3

<u>Claim</u>: Coxeter complex of type A_{n-1}.

<u>Exercise</u>

Σ^A = Coxeter complex type A_{n-1}

Then Σ^A = barycentric subdivison of bdry of simplex

= poset of nontrivial proper subsets of $[n]$.

Check B0: Let Σ be an apt in Δ_n, associated to frame $L_1 \oplus \cdots \oplus L_n = V$

Then
$$\begin{array}{ccc} \Sigma^A & \xrightarrow{\varphi} & \Sigma \\ I \subseteq [n] & \longmapsto & \bigoplus_{i \in I} L_i \end{array}$$

Let C be chamber $\underset{V_1}{\underset{\shortparallel}{L_1}} \subseteq \underset{V_2}{\underset{\shortparallel}{L_1 \oplus L_2}} \oplus \cdots$

Then φ has inverse

$$\begin{array}{ccc} \Sigma^A & \xleftarrow{\varphi_C} & \Sigma \\ \left\{ i \mid \begin{array}{l} U \cap V_i \\ > U \cap V_{i-1} \end{array} \right\} & \longleftarrow\!\shortmid & U \end{array}$$

identification of $[n]$ with $\{L_1, \ldots, L_n\}$ given by choice of chamber C.

Check B2

Fact B2 is equivalent to

B2″ Σ, Σ' apts with common chamber C. Then $\exists$ iso $\Sigma \to \Sigma'$ fixing every simplex in $\Sigma \cap \Sigma'$

But φ only depended on chamber C (not choice of frame)

so if $C \in \Sigma, \Sigma'$ then

$$\Sigma \xrightarrow{\varphi_C} \Sigma^A \xrightarrow{\varphi} \Sigma'$$

is the desired iso.

B1 Given flags $U_1 \subseteq U_2 \subseteq \ \subseteq U_{n-1}$ C

$U_1' \subseteq U_2' \subseteq \cdots U_{n-1}'$ C'

want frame $L_1 \oplus L_2 \oplus \ \oplus L_n = U$ compatible with both

Define <u>Jordan-Holder permutation</u> $\pi(C, C')$

$\pi : [n] \longrightarrow [n]$

$\pi(i) = j$ for unique j such that

$$U_{i-1}' + (U_i' \cap U_k) = \begin{cases} U_{i-1}', & k < j \\ U_i', & k \geq j. \end{cases}$$

[Equivalently, $U_1 \subseteq U_2 \subseteq \cdot \subseteq U_{n-1}$ induces a filtration on the 1-dim^{e} vector space $\frac{U_i'}{U_{i-1}'}$. j is the index the dimension jumps 0 to 1.]

<u>Exercise</u>: $\pi(C', C)$ is inverse to $\pi(C, C')$.
$\Rightarrow$ it is a permutation.

<u>Exercise</u> If $\pi(i) = j$, then the canonical maps surject:

$$\frac{U_j}{U_{j-1}} \longleftarrow U_i' \cap U_j \longrightarrow \frac{U_i'}{U_{i-1}'}$$

Choose L_j to be any line in $U_i' \cap U_j$ with nontrivial image in U_i'/U_{i-1}' and U_j/U_{j-1}.

<u>Fact</u> This permutation is the Weyl distance.

$$\delta(C, C') = \pi(C, C').$$

Complete system of apartments

Thm / Defn Δ building

The union of any family of apartment systems is again an apartment system.

The maximal apartment system is the complete apartment system. $\mathcal{A}$.

Fact The complete system of apartments is the union of all subcomplexes Σ_G in Δ isomorphic to the Coxeter complex for assoc. gp G.

Solomon - Tits

Thm Δ - building

C - chamber in Δ

If Δ is spherical of rank n, then

$$|\Delta| \simeq \vee S^{n-1}$$

with one sphere for every apt containing C.

If Δ is not spherical, then $|\Delta|$ is contractible.

Pf Outline

Build Δ inductively

- begin with C
- glue in all chambers distance 1 from C
- glue in all chambers distance 2 from C

etc.

When $d <$ diam(C), chambers of distance d are glued down along contractible subset of their boundary. Result is contractible.

When $d = \mathrm{diam}(C)$, each chamber of distance d is glued along its entire boundary. Result is $(n-1)$-sphere.