Michigan Representation Stability Seminar
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Quillen's approach to homological stability

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These notes follow two lectures in the Michigan Representation Stability Seminar from April 2020.

Lecture 1: Spectral sequences and semi-simplicial objects

1 A review of spectral sequences

A *spectral sequence* is a computational tool that can be viewed as a generalization of the familiar long exact sequences from algebraic topology.
For example,

- Just as there is a long exact sequence of a pair (X, A), there is a spectral sequence associated to a filtration of a space
$$\varnothing \subseteq X_0 \subseteq X_1 \subseteq \cdots \subseteq X_N = X.$$
- Just as there is the Mayer-Vietoris long exact sequence, there is a spectral sequence associated to an open cover $\{\mathcal{U}_i\}$ of a space X.

As with the long exact sequences in algebraic topology, we can sometimes use spectral sequences to do (co)homology calculations simply using formal algebraic properties of the spectral sequences, without needing to understand its construction or determine the individual maps.

Homology spectral sequences

A spectral sequence is a "book" consisting of a sequence of *pages* (or *sheets*). These are typically denoted E^r (or E_r for a cohomology spectral sequence), for $r = 0, 1, 2, \ldots$. Each page is a bigraded abelian group with a differential structure. Concretely, each page has

- a 2D array of groups (or rings, or algebras) $E^r_{p,q}$, with $(p, q) \in \mathbb{Z}^2$,
- a map $d^r : E^r \to E^r$ satisfying $(d^r)^2 = 0$, called the *differential*.

The differentials d^r give E^r the structure of a chain complex. The page E^{r+1} is the homology of the complex (E^r, d^r), in the sense that

$$E^{r+1}_{p,q} = \frac{\text{kernel of } d^r \text{ at } E^r_{p,q}}{\text{image of } d^r \text{ in } E^r_{p,q}}.$$

In particular the group $E^{r+1}_{p,q}$ is always a subquotient of $E^r_{p,q}$.

Unfortunately, although the complex (E^r, d^r) determines the groups $E^{r+1}_{p,q}$, it does not determine the differential d^{r+1}.

For the spectral sequences we will consider today,

- the group $E^r_{p,q}$ is only possibly nonzero for $p, q \geq 0$. Such a spectral sequence is called a *first quadrant spectral sequence*.
- The differentials satisfy
$$d^r : E^r_{p,q} \longrightarrow E^r_{p-r,q+r-1}.$$

See Figure 1.

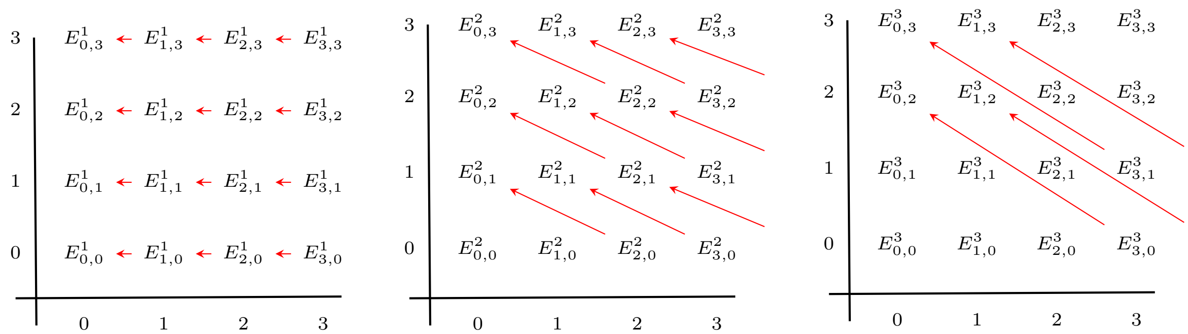


Figure 1: The pages E^1, E^2, and E^3.

Exercise 1. (The structure of a homology spectral sequence). Let $E^*_{*,*}$ be a first quadrant homology spectral sequence.

(a) Verify that the groups $E^r_{p,0}$ along the bottom $q = 0$ row are always subgroups of the groups $E^s_{p,0}$ on the previous pages $s < r$. The maps
$$E^r_{p,0} \hookrightarrow E^{r-1}_{p,0} \hookrightarrow \cdots \hookrightarrow E^3_{p,0} \hookrightarrow E^2_{p,0}$$
are called *edge maps*.

(b) Verify that the groups $E^r_{0,q}$ along the bottom $q = 0$ row are always quotients of the groups $E^s_{0,q}$ on the previous pages $s < r$. The maps
$$E^2_{0,q} \twoheadrightarrow E^3_{0,q} \twoheadrightarrow \cdots \twoheadrightarrow E^{r-1}_{0,q} \twoheadrightarrow E^r_{0,q}$$
are also called *edge maps*.

(c) Show that, for fixed (p, q), there is equality $E^{r+1}_{p,q} = E^r_{p,q}$ if and only if both the incoming and outgoing differentials at $E^r_{p,q}$ are the zero map.

Convergence

Suppose for each (p, q), the groups $E^r_{p,q}$ eventually stabilize, in the sense that (for all r sufficiently large, depending on p and q),
$$E^r_{p,q} = E^{r+1}_{p,q} = E^{r+2}_{p,q} = \cdots$$
In this case, we write $E^\infty_{p,q}$ for the stable groups, and call the bigraded object $E^\infty = \{E^\infty_{p,q}\}_{p,q}$ the *limit* or *abutment* of the spectral sequence. This limit page does not have a differential.

Observe that, if we have a first quadrant spectral sequence, then at at any fixed point (p, q), for r sufficiently large, either the domain or the codomain of both differentials d^r to or from $E^r_{p,q}$ will be zero. Thus, the spectral sequence must stabilize at every point (p, q).

In general the sequence of groups $\{E^r_{p,q}\}_r$ stabilizes at a page r that depends on (p, q). If there is some r such that $E^r_{p,q} = E^\infty_{p,q}$ for all p and q, then we say that the spectral sequence *degenerates* on page E_r.

> **Exercise 2. (The structure of a homology spectral sequence).** Let $E^*_{*,*}$ be a first quadrant homology spectral sequence. For a given (p, q), on what page r must the group $E^*_{p,q}$ stabilize?

The Leray–Serre spectral sequence (homology version)

Let G be an abelian group. Let

$$F \longrightarrow X \xrightarrow{\pi} B$$

be a fibration with a path-connected base space B. Then there is a *monodromy* action of $\pi_1(B)$ on the homology $H_*(F; G)$ of the fibre F. We use the notation $\mathscr{H}_*(F; G)$ when viewing this homology group as a $G[\pi_1(B)]$-module.

Theorem I. (The homology Leray–Serre spectral sequence). *Let G be an abelian group. Given a fibration*

$$F \longrightarrow X \xrightarrow{\pi} B$$

with a path-connected base space B, there is a spectral sequence $\{E^r_{p,q}, d_r\}$, called the Leray–Serre spectral sequence, with the following properties. The differentials satisfy

$$d^r : E^r_{p,q} \longrightarrow E^r_{p-r,q+r-1}.$$

The homology $\mathscr{H}_q(F)$ is a $G[\pi_1(B)]$–module, and the E^2 page is the bigraded algebra of cohomology groups with twisted coefficients

$$E^2_{p,q} = H_p(B; \mathscr{H}_q(F; G)).$$

The spectral sequence converges to the cohomology groups

$$H_{p+q}(X; G)$$

in the sense that there is some filtration of $H_k(X; G)$

$$0 = F^{-1}_k \subseteq F^0_k \subseteq \cdots \subseteq F^k_k = H_k(X; G)$$

such that the limiting groups $E^\infty_{p,q}$ are the associated graded pieces

$$E^\infty_{p,q} = F^p_{p+q}/F^{p-1}_{p+q}.$$

The E^∞ page is shown in Figure 2.

We say that the spectral sequence *converges* to $H_{p+q}(X; G)$ and write

$$E^2_{p,q} \implies H_{p+q}(X; G).$$

Convergence is implicitly in the sense of the associated graded groups.

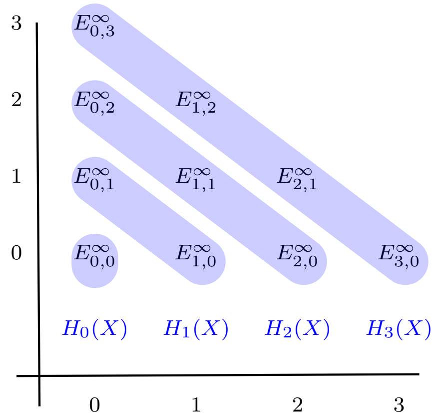


Figure 2: The limit of the Serre spectral sequence.

Remark II. (Twisted coefficients on E^2). When the $\pi_1(B)$-action on $\mathscr{H}_q(F)$ is trivial – for example, if B is simply connected – then the E^2 page is simply the groups

$$E^2_{p,q} = H_p(B; H_q(F;G))$$

with ordinary coefficients.

Remark III. (Recovering $H_*(X)$). In general, knowing the quotient groups $E^\infty_{p,q} = F^p_{p+q}/F^{p-1}_{p+q}$ is not enough to reconstruct the cohomology groups $H_*(X)$; we can only determine these groups "up to extensions". Consider, for example, the simplest example: If $G = \mathbb{Z}$, and $E^\infty_{0,1} \cong E^\infty_{1,0} \cong \mathbb{Z}/2\mathbb{Z}$, then the theorem states that $H_1(X)$ fits into a short exact sequence

$$0 \longrightarrow E^\infty_{0,1} \longrightarrow H_1(X) \longrightarrow E^\infty_{1,0} \longrightarrow 0$$

but the theorem does not distinguish between the two resultant possibilities, whether $H_1(X)$ is $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ or $\mathbb{Z}/4\mathbb{Z}$.

Exercise 3. (The limit of the Leray–Serre spectral sequence). Let $E^r_{p,q}$ be the Leray-Serre spectral sequence for a fibration $F \longrightarrow X \xrightarrow{\pi} B$.

(a) Suppose that $G = \mathbb{F}$ is a field. Show that

$$H_k(X;\mathbb{F}) = \bigoplus_{p+q=k} E^\infty_{p,q}.$$

In this case there are no extension problems.

(b) Let $G = \mathbb{Z}$. Suppose that, for some k, all the groups $E^\infty_{p,q}$ with $p+q=k$ are free abelian. Show that

$$H_k(X) = \bigoplus_{p+q=k} E^\infty_{p,q}.$$

Exercise 4. (The relationship to the product $X \times Y$). Let $\mathbb{F}$ be a field. Let F, B be path-connected topological spaces and consider a fibration

$$F \longrightarrow X \xrightarrow{\pi} B$$

with trivial monodromy.

(a) Use the Künneth formula to prove that, if all the differentials d^r are identically zero for $r \geq 2$, then
$$H_*(X) = H_*(F \times B).$$

(b) Explain the sense in which (with no assumptions on the differentials) the homology of $H_*(X)$ must be 'smaller' than the homology of $H_*(F \times B)$..

We can view the Serre spectral sequence as measuring how far the fibration π is from being the trivial fibration.

Exercise 5. (Contractible base and fibres). Let
$$F \longrightarrow X \xrightarrow{\pi} B$$
be a fibre bundle.

(a) Suppose that B is contractible. Use the Leray-Serre spectral sequence to verify that $H_*(X) = H_*(F)$. In fact, all fibre bundles over a contracible space are trivial bundles.

(b) Suppose alternatively that F is contractible. Show that $H_*(X) = H_*(F)$. In fact, if the spaces have the homotopy type of CW-complexes, then in this case π must be a homotopy equivalence.

We can view the Serre spectral sequence as measuring how far the fibration π is from being the trivial fibration.

An application of the Leray–Serre spectral sequence

As a toy example to illustrate the use of the Leray–Serre spectral sequence, we will address the following question. Consider a fibre bundle
$$S^3 \longrightarrow X \longrightarrow S^4$$
with total space X. What are the possibilities for the homology of X? The E_2 page of the Leray–Serre spectral sequence is shown in Figure 3. The only possibly-nonzero differential, a d_4 differential, is shown.

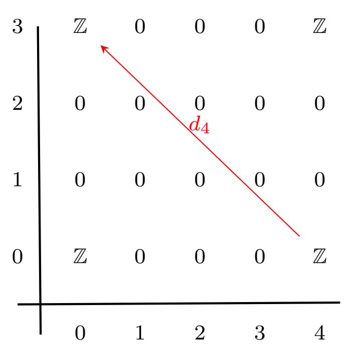


Figure 3: The E^2 page of the Leray–Serre spectral sequence for a fibration $S^3 \to X \to S^4$

Thus the spectral sequence collapses on the E_5 page, and the homology of X is given by

$$H_k(X) = \begin{cases} \mathbb{Z}, & k = 0 \\ \mathbb{Z}/\mathrm{im}(d_4), & k = 3 \\ \ker(d_4), & k = 4 \\ \mathbb{Z}, & k = 7. \end{cases}$$

In the case of the generalized Hopf fibration

$$S^3 \longrightarrow S^7 \longrightarrow S^4$$

the nonzero d_4 differential must be an isomorphism, and kill the two copies of $\mathbb{Z}$ in homological degrees 3 and 4.

Exercise 6. As another toy application of the Leray–Serre spectral sequence, use the homology of S^1 and S^∞ and the fibration

$$S^1 \to S^\infty \to \mathbb{C}P^\infty$$

to compute the homology of $\mathbb{C}P^\infty$. You may assume that $\mathbb{C}P^\infty$ is simply connected.

2 Semi-simplicial objects

Definition IV. The *semi-simplicial category* Δ_+ is the category with

- objects: $[n]_0 = \{0, 1, 2, 3, \ldots, n\}$ for $n \in \mathbb{Z}_{\geq 0}$
- morphisms: strictly increasing maps.

See Figure 4. For $i = 0, \ldots, n+1$, write δ_i for the strictly increasing map $[n]_0 \to [n+1]_0$ that 'misses' the element i.

$$[0]_0 \longrightarrow [1]_0 \rightrightarrows [2]_0 \ \substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow} \ [3]_0 \ \substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow \\ \longrightarrow} \cdots$$

Figure 4: The category Δ_+

Exercise 7. Verify the relation $\delta_j \circ \delta_i = \delta_i \circ \delta_{j-1}$ for all $i < j$.

Definition V. A *semi-simplicial object* in a category $\mathcal{C}$ is a functor

$$X_\bullet : \Delta_+^{op} \longrightarrow \mathcal{C}$$

The object $X_p := X([p]_0)$ is called the *p-simplices*. Let

$$d_i : X_p \to X_{p-1}, \qquad i = 0, \ldots, p$$

denote the functorial image of the map δ_i. We call d_i the *face maps*.

It turns out that the relation given in Exercise 2 generates all relations among the morphisms δ_i, and so the relations among the face maps

$$d_i \circ d_j = d_{j-1} \circ d_i \quad \text{for all } i < j$$

are the defining relations of a semi-simplicial object in $\mathcal{C}$.

Definition VI. A map of semi-simplicial objects $F : X_\bullet \to Y_\bullet$ in a category $\mathcal{C}$ is a natural transformation F of functors.

In other words, a map of semi-simplicial objects is a sequence of maps of p-simplices $F_p : X_p \to Y_p$ that commute with the face maps,

$$F_p \circ d_i = d_i \circ F_{p+1}.$$

2.1 Semi-simplicial sets

Let $\underline{\text{Set}}$ be the category of sets, and consider a semi-simplicial set

$$X_\bullet : \Delta_+^{op} \longrightarrow \underline{\text{Set}}$$

We may view the elements of X_p as topological p-simplices, each with their faces ordered 0 through p. Then we can view the face map d_i as providing "gluing instructions" on how to identify the i^{th} face of each p-simplex with a $(p-1)$-simplex, to assemble these simplices into a CW complex.

Geometric realization

Let Δ^p denote a topological p-simplex, and let $d^i : \Delta^{p-1} \hookrightarrow \Delta^p$ denote the inclusion of the i^{th} face.

Given a semi-simplicial set $X_\bullet$, we define its *geometric realization* to be the following topological space $||X_\bullet||$ with the quotient topology,

$$||X_\bullet|| = \left(\coprod_{p=0}^{\infty} X_p \times \Delta^p \right) \Big/ \sim \qquad \text{with } (\sigma, d^i t) \sim (d_i \sigma, t) \text{ for all } \sigma \in X_p, t \in \Delta^{p-1}.$$

Exercise 8. Let $X_\bullet$ be a semi-simplicial set with

$$X_0 = \{v_0, v_1, v_2\}$$
$$X_1 = \{\ell_0, \ell_1, \ell_2\}$$
$$X_2 = \{T_0, T_1\}$$
$$X_p = \varnothing \quad \text{for } p > 2$$

Determine how the face maps should be defined so that the geometric realization of $X_\bullet$ is the 2-sphere shown in Figure 5.

Remark VII. A semi-simplicial set is more general than a simplicial complex; in the geometric realization a simplex need not be uniquely specified by its vertices.

Given a semi-simplicial set $X_\bullet$, the homology of $||X_\bullet||$ is computed by the chain complex with p-chains $\mathbb{Z}X_p$ and differential given by the alternating sum of the face maps:

$$d = \sum_{i=0}^{p} (-1)^i d_i : \mathbb{Z}X_p \to \mathbb{Z}X_{p-1}.$$

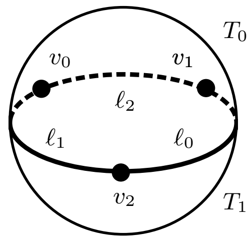


Figure 5: A semi-simplicial sphere

An aside: the standard simplex

Let $\mathrm{Hom}_{\Delta_+}(\bullet, n)$ denote the functor

$$\begin{aligned}\mathrm{Hom}_{\Delta_+}(\bullet, n) : (\Delta_+)^{op} &\longrightarrow \underline{\mathrm{Set}} \\ [p]_0 &\longmapsto \mathrm{Hom}_{\Delta_+}([p]_0, [n]_0)\end{aligned}$$

and a contravariant action of Δ_+ morphisms by postcomposition. We will see the sense in which we can think of $\mathrm{Hom}_{\Delta_+}(\bullet, n)$ as an n-simplex Δ^n.

Exercise 9. ($\mathrm{Hom}_{\Delta_+}(\bullet, n)$ as an n-simplex).

(a) Verify that the $\mathrm{Hom}_{\Delta_+}(\bullet, n)$ is in fact a semi-simplicial set, by describing the action of the morphisms and checking functoriality.

(b) Show that

$$||\mathrm{Hom}_{\Delta_+}(\bullet, n)|| \simeq \Delta^n,$$

with the identity map id_n corresponding to the top cell, and the other face maps corresponding to the faces.

The semi-simplicial set $\mathrm{Hom}_{\Delta_+}(\bullet, n)$ is sometimes called the *standard n-simplex* and denoted $\Delta^n_\bullet$.

Let $X_\bullet$ be a semi-simplicial set. By the Yoneda Lemma, there is a natural bijection between X_n and the set of natural transformations:

$$\mathrm{Nat}\Big(\mathrm{Hom}_{\Delta_+}(\bullet, n) \to X_\bullet\Big) \;\cong\; X_n$$
$$\begin{bmatrix} id_n & \mapsto & x \\ f & \mapsto & f(x) \end{bmatrix} \longleftrightarrow x$$

This bijection is an analogue of the familiar identification of R-modules M

$$\mathrm{Hom}_R(R, M) \;\cong\; M$$
$$\begin{bmatrix} 1 & \mapsto & m \\ r & \mapsto & r \cdot m \end{bmatrix} \longleftrightarrow m$$

We may therefore view X_n as being, in a natural sense, the set of semi-simplicial maps

$$\Delta^n_\bullet \longrightarrow X_\bullet$$

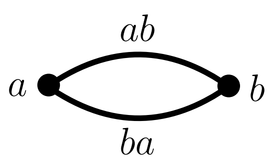
Example VIII. (The complex of injective words).
Fix $n \in \mathbb{Z}_{\geq 1}$. The *complex of injective words* is a semi-simplicial set $\text{Inj}_\bullet = \text{Inj}_\bullet(A)$ associated to an alphabet A, defined as follows.

- p-simplices Inj_p are words on $(p+1)$ **distinct** letters in A
- face maps $d_i : \text{Inj}_p \to \text{Inj}_{p-1}$ deletes the i^{th} letter of a word (counting from 0).

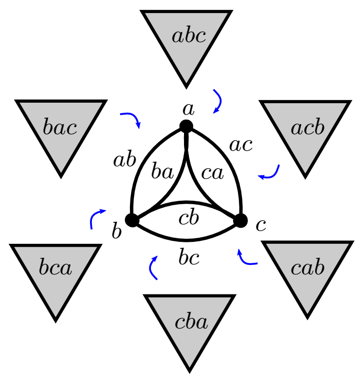
For example,

$$d_0(abc) = bc \qquad d_1(abc) = ac \qquad d_2(abc) = ab.$$

Then $||\text{Inj}_\bullet(\{a,b\})|| \simeq S^1$ and $||\text{Inj}_\bullet(\{a,b,c\})|| \simeq S^2 \vee S^2$, as in Figure 6.



(a) $||\text{Inj}_\bullet(\{a,b\})||$



(b) $||\text{Inj}_\bullet(\{a,b,c\})||$

Figure 6: The geometric realization of the complex of injective words

It is a theorem of Farmer [Fa] that, in general, $||\text{Inj}_\bullet(A)||$ is homotopy equivalent to a wedge of spheres of dimension $|A|-1$. We will return to this fact in the next lecture.

Semi-simplicial spaces and the associated spectral sequence

Let $X_\bullet$ denote a semi-simplicial space, that is, a functor to topological spaces

$$X_\bullet : \Delta_+^{op} \to \underline{\text{Top}}.$$

We now have, for each p, a topological space X_p parameterizing the p-simplices of $X_\bullet$. We can again define the geometric realization by the quotient topology

$$||X_\bullet|| = \left(\coprod_{p=0}^{\infty} X_p \times \Delta^p \right) \Big/ \sim \qquad \text{with } (\sigma, d^i t) \sim (d_i \sigma, t) \text{ for all } \sigma \in X_p, t \in \Delta^{p-1}$$

and now the topology on $||X_\bullet||$ incorporates both the simplicial structure and the topology of the spaces X_p.

From a semi-simplicial space, we obtain a first quadrant spectral sequence, with

$$E^1_{p,q} = H_q(X_p)$$

converging to the associated graded groups of a filtration of the homology groups $H_{p+q}(||X_\bullet||)$. The d^1 differentials

$$E^1_{p-1,q} \xleftarrow{d^1} E^1_{p,q}$$

are given by the alternating sum of the maps induced by the face maps. By abuse of notation, we also denote these induced maps by d_i,

$$d^1 = \sum_{i=0}^{p} (-1)^i d_i$$

See Figure 7. In general d^r satisfies $d^r : E^r_{p,q} \longrightarrow E^r_{p-r,q+r-1}$.

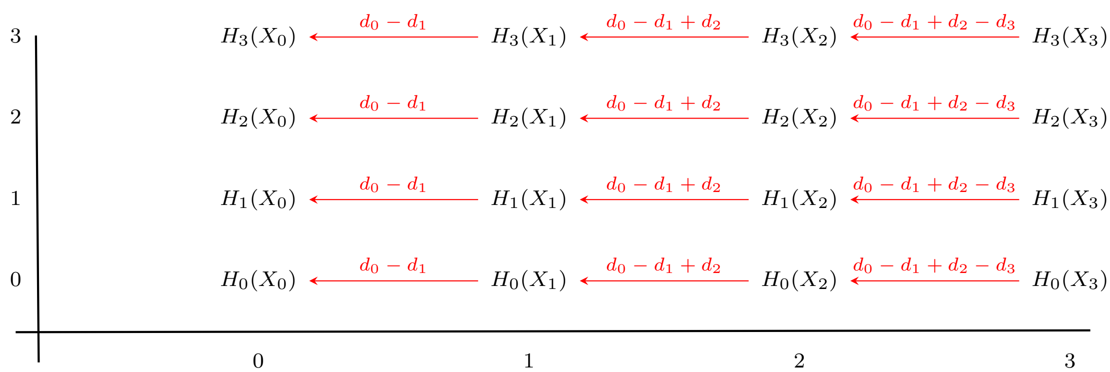


Figure 7: The E_1 page of the spectral sequence for a semi-simplicial space.

Example IX. As a toy example, we will use this spectral sequence to analyze the semi-simplicial space defined as follows.

- $X_0 = *$
- $X_1 = S^1$
- $X_p = \varnothing$ for $p > 1$.

The face maps $d_0, d_1 : S^1 \to *$ are necessarily the constant map.

The E^1 page of the associated spectral sequence is shown in Figure 8. Since the maps $d_1 = d_0$, the only possibly-nonzero d^1 differential is in fact zero.

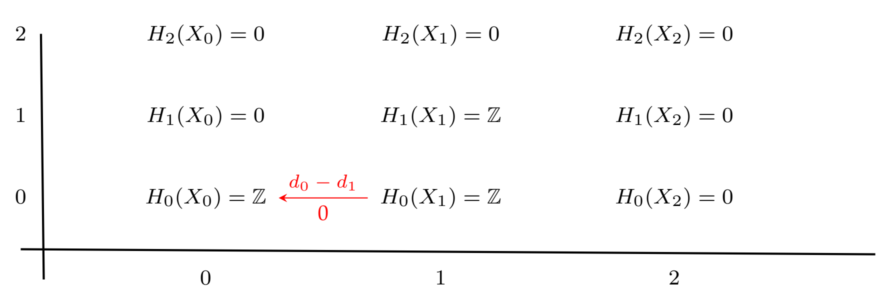


Figure 8: The page E_1.

Observe that, for any higher differential d^r, either the domain or the range must be zero. Hence the sequence collapses on page E_1; see Figure 9.

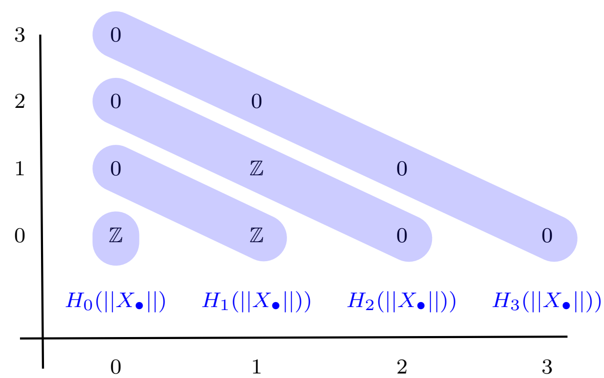


Figure 9: The page E^∞

There are no extension problems, and we find that the limit

$$H_k(||X_\bullet||) = \begin{cases} \mathbb{Z}, & k = 0 \\ \mathbb{Z}, & k = 1 \\ \mathbb{Z}, & k = 2 \\ 0, & k > 2 \end{cases}$$

This is consistent with a direct construction of $||X_\bullet||$, shown in Figure 10.

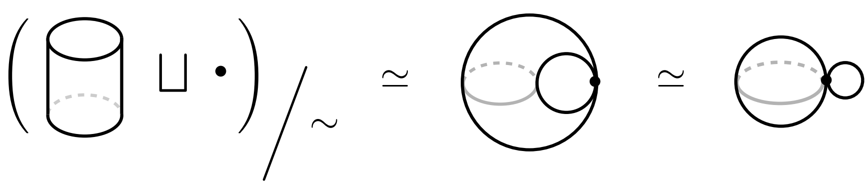


Figure 10: $||X_\bullet|| = \left(\coprod_{p=0}^{\infty} X_p \times \Delta^p\right) \big/ \sim$

Exercise 10. Consider the semi-simplicial space with $X_0 = \{0, 1\}$, $X_1 = S^1$, and the face map $d_i : X_1 \to X_0$ given by the constant map to the vertex i. Compute the E^1 and E^∞ pages of the associated spectral sequence. Compare the results to a direct computation of $H_*(||X_\bullet||)$.

Exercise 11. Let $X_\bullet$ be a semi-simplicial space with the property that the spaces X_p have the discrete topology for all p. Compute the E^1 page of the associated spectral sequence, and show that this spectral sequence collapses on page E^1. Verify that the E^∞ determines the homology groups $H_*(||X_\bullet||)$, and that these homology groups coincide with the homology groups of $X_\bullet$ when viewed as a semi-simplicial set.

Lecture 2: Quillen's proof of homological stability

3 Homological stability

Definition X. For $n \geq 1$ let G_n be a family of groups (or topological spaces) with inclusions

$$G_1 \xrightarrow{\varphi_1} G_2 \xrightarrow{\varphi_2} G_3 \xrightarrow{\varphi_3} \cdots$$

Then the family $\{G_n\}_{n\geq 1}$ is called *homologically stable* if, for each $k \geq 0$,

$$H_k(G_n) \xrightarrow{(\varphi_n)_*} H_k(G_{n+1})$$

is an isomorphism for all $n >> k$.

In other words, for each k, the k–dimensional structure of the groups G_n (at least as captured by their homology) eventually stabilizes as n grows.

Homological stability is an important and pervasive phenomenon in algebraic topology. Some examples of groups and spaces exhibiting this behaviour:

- (Nakaoka 1960s) symmetric groups S_n
- (Arnold 1968, Cohen 1972) braid groups Br_n
- (Quillen unpublished, Maazen 1979, van der Kallen 1980, Charney 1979) linear groups like GL_n
- (Harer, 1985) mapping class groups of surfaces (increasing genus *or* # boundary components)
- (Hatcher 1995, Hatcher–Vogtmann, Hatcher–Vogtmann–Wahl) the (outer) automorphism group of a free group, $\mathrm{Aut}(F_n)$ and $\mathrm{Out}(F_n)$
- (McDuff 1975, Segal 1979) configuration spaces $C_n(M)$ of an open manifold M

Quillen's approach to homological stability

Quillen described a strategy to prove homological stability for a sequence of groups or spaces G_n by using induction on a sequence of spectral sequences that relate the homology of G_n to the homology of smaller groups G_{n-i}.

We will see a version of this argument for a family of groups G_n. The strategy depends on the existence of G_n-spaces with certain properties. One of those properties is that it must be highly connected, in the sense of the following definition

Definition XI. A topological space Y is called (-1)*-connected* if it is nonempty, and 0*-connected* if it is path-connected. More generally, for $n \in \mathbb{Z}_{\geq 0}$, the space is n*-connected* if it is nonempty and

$$\pi_i(Y) = 0 \qquad \text{for all } 0 \leq i \leq n.$$

Theorem XII. (Quillen's proof of homological stability). *Let $\{G_n\}$ be a family of groups, $n \in \mathbb{Z}_{\geq 1}$, with inclusions*

$$\varphi_n : G_n \to G_{n+1}.$$

For each n let $(Y_n)_\bullet$ be a semi-simplicial set with a G_n action, satisfying the following properties.

- *The spaces $||(Y_n)_\bullet||$ are highly connected, that is, they are r_n–connected for some function r_n that tends to infinity. For concreteness we take $r_n = (n-2)$.*
- *For each p, the group G_n acts transitively on the set of p-simplices $(Y_n)_p$.*
- *The stabilizer stab(σ_p) of a p-simplex σ_p is isomorphic to G_{n-p-1}, and is conjugate to the distinguished copy $G_{n-p-1} \subseteq G_n$ determined by the inclusion maps φ_i.*
- *The stabilizer stab(σ_p) fixes σ_p pointwise for each simplex σ_p.*
- *The inclusion of a face $\tau_q \hookrightarrow \sigma_p$ induces an inclusion of groups $G_{n-p-1} \hookrightarrow G_{n-q-1}$ conjugate to the distinguished inclusion.*

Then the sequence $\{G_n\}$ is homologically stable.

Before we begin the proof, we introduce some notation. Let G be a group. Let A be a right G-set and B a left G-set. We write $A \times_G B$ to denote the quotient of $A \times B$ by the identification $(a \cdot g, b) \sim (a, g \cdot b)$ for $g \in G$. We will use the following lemma:

Lemma XIII. *Let X be a right G-set. Then $X \times_G (G/H) \cong X/H$.*

Proof. Let $*$ denote a point.

$$\begin{aligned} X \times_G (G/H) &\cong X \times_G (G \times_H *) \\ &\cong (X \times_G G) \times_H * \\ &\cong X \times_H * \\ &\cong X/H. \end{aligned}$$

□

> **Exercise 12. (Balanced products).** Verify all the isomorphisms in the proof of Lemma XIII.

We will now see Quillen's proof. To be concrete, we let $r_n = n-2$. We will prove that $(\varphi_n)_*$ induces an isomorphism on degree-k homology for all $n \geq 2k$.

Proof of Theorem XII. Let EG_n denote a contractible space with a free right G_n–action. (Since G_n is discrete, we can think of EG_n as the universal cover for a $K(G_n, 1)$ space.) Define a semi-simplicial space

$$(X_n)_\bullet = EG_n \times_{G_n} (Y_n)_\bullet.$$

We note that, if $||(Y_n)_\bullet||$ were contractible, then this would be a model for the classifying space BG_n of G_n, and its homology would coincide with the homology of G_n. Under the current assumptions, the homology of $(X_n)_\bullet$ agrees with the homology of G_n in a range:

$$H_k(||(X_n)_\bullet||) \cong H_k(G_n) \quad \text{for } k \leq r_n = n-2.$$

By assumption, $(Y_n)_p \cong G_n/G_{n-p-1}$ as a G_n-set. Thus by our lemma,

$$\begin{aligned} (X_n)_p &\cong EG_n \times_{G_n} (Y_n)_p \\ &\cong EG_n \times_{G_n} (G_n/G_{n-p-1}) \\ &\cong EG_n/G_{n-p-1} \end{aligned}$$

so $(X_n)_p$ is a model for the classifying space for G_{n-p-1}. In particular,

$$H_k((X_n)_p) = H_k(G_{n-p-1}) \quad \text{for all } k.$$

Let $E^r_{p.q}(n)$ denote the spectral sequence associated to the semi-simplicial space $(X_n)_\bullet$. By the above discussion, this spectral sequence satisfies the following.

- The E^1 page satisfies

$$\begin{aligned} E^1_{p,q}(n) &\cong H_q(EG_n \times_{G_n} (Y_n)_p) \\ &\cong H_q(G_{n-p-1}) \end{aligned}$$

- The spectral sequence converges to the groups

$$H_{p+q}(||(X_n)_\bullet||) \cong H_{p+q}(G_n) \quad \text{for } (p+q) \leq r_n.$$

- The d^1 differential is the alternating sum of the maps induced by the face maps

$$d^1 = \sum_{i=0}^{p} (-1)^i d_i$$

  Because the face maps are all conjugate to the map φ_{n-p-1} by assumption, they induce the same map $(\varphi_{n-p-1})_*$ on the homology. Thus

$$d^1 = \begin{cases} 0, & p \text{ odd} \\ (\varphi_{n-p-1})_*, & p \text{ even.} \end{cases}$$

The E^1 page is shown in Figure 11.

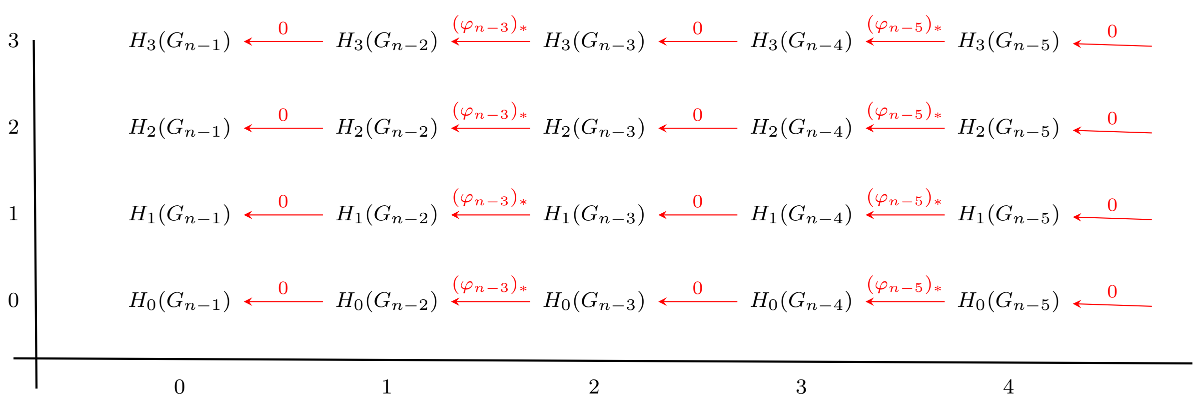


Figure 11: The E_1 page of the spectral sequence for $(X_n)_\bullet$.

We may now proceed by induction on n. Our goal is to prove that, for a given value of n, the induced maps $(\varphi_n)_* : H_k(G_n) \to H_k(G_{n+1})$ are isomorphisms for all $k \leq \frac{n}{2}$.

For the base case, when $n = 1$, it suffices to observe that the map

$$(\varphi_1)_* : H_0(G_1) \to H_0(G_2)$$

is necessarily an isomorphism.

For the inductive step, we fix n and we assume that the result holds for all $i < n$. Now consider the $E^1(n+1)$ page of the spectral sequence. The nonzero d^1 differentials

$$d^1 = (\varphi_{n-p-1})_*$$

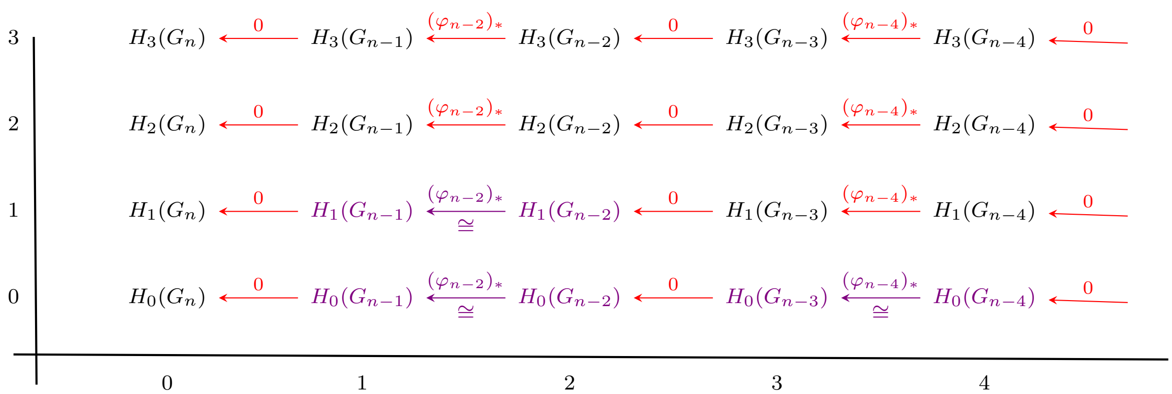


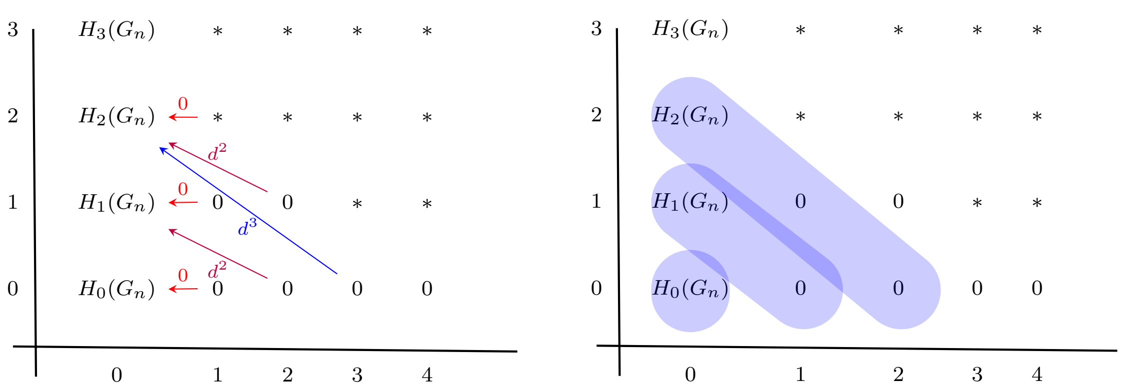
Figure 12: The inductive step on the $E_1(n+1)$ page illustrated for $n = 5$. By inductive hypothesis, the maps coloured violet are isomorphisms.

are, by inductive hypothesis, isomorphisms whenever $2q \leq n - p - 1$. These d^1 differentials therefore kill their domain and codomains, and both groups are zero on the E_3 page. See Figure 12.

As a result, all higher differentials to and from the groups

$$H_0(G_n), H_1(G_n), \ldots, H_{\lfloor \frac{n}{2} \rfloor}(G_n)$$

in the $p = 0$ column are necessarily zero, as in Figure 13a.



(a) The $E^r(n+1)$ page illustrated for $n = 5$.

(b) The limiting values with $p + q \leq \frac{n}{2}$, illustrated for $n = 5$.

Thus for $q \leq \frac{n}{2}$, we find that $E^{\infty}_{0,q} = H_q(G_n)$. And, this $p = 0$ column contains the only nonzero groups on or under the diagonal $p + q \leq \frac{n}{2}$, hence we have isomorphisms to the limit, which in this range is equal to the groups $H_q(G_{n+1})$. See Figure 13b. It can be checked that the resultant isomorphisms

$$H_q(G_n) \cong H_q(G_{n+1})$$

are induced by the maps $(\varphi_n)_*$. This concludes the inductive step, and the proof. □

Exercise 13. (Checking some proof details).

(a) Explain why the maps $(\varphi_n)_* : H_0(G_n) \to H_0(G_{n+1})$ are necessarily isomorphisms in homological degree zero.

(b) Verify that, under the inductive hypothesis, all differentials to and from the groups $E^r_{0,q}(n+1) = H_q(G_n)$ are zero for all $q \leq \frac{n}{2}$.

(c) Verify that, under the inductive hypothesis, the groups $E^2_{p.q}(n+1)$ vanish for $p+q \leq \frac{n}{2}$ if $p > 0$.

(d) Verify that the spectral sequence $E^r_{p,q}(n+1)$ converges to $H_q(G_{n+1})$ in the range $p+q \leq \frac{n}{2}$.

3.1 Homological stability for the symmetric groups

Theorem XIV (Nakoaka, 60's). *Let φ_n denote the standard inclusion of the symmetric group*

$$\varphi_n : S_n \to S_{n+1}.$$

Then

$$(\varphi_n)_* : H_k(S_n) \longmapsto H_k(S_{n+1})$$

is an isomorphism for all $n \geq 2k$.

To prove this theorem using Quillen's method, we can take $(Y_n)_\bullet$ to be the complex of injective words from Example VIII, on the alphabet $[n] = \{1, 2, 3 \ldots, n\}$. Recall that Farmer [Fa] proved that $(Y_n)_\bullet$ is $(n-2)$–connected.

Exercise 14. (Checking the properties of Inj$_\bullet([n])$). Verify the following.

(a) The symmetric group S_n acts transitively on the set of p-simplices Inj$_p([n])$ for each p.

(b) A p-simplex corresponds to a word $w = a_0 a_1 \ldots a_p$ on $(p+1)$ distinct letters in $[n]$. Show that the stabilizer of w is isomorphic to S_{n-p-1}, and conjugate to the distinguished copy of $S_{n-p-1} \subseteq S_n$.

(c) Verify that the stabilizer of the simplex w fixes the simplex pointwise by checking that it fixes its $(p+1)$ vertices pointwise.

(d) Describe what it means for a word u on $(q+1)$ letters to be a face of w. Verify that the inclusion of this face induces an inclusion of groups $S_{n-p-1} \hookrightarrow S_{n-q-1}$ conjugate to the distinguished inclusion.

Homological stability follows from Exercise 3.1.

We will see in the next exercise that there are models for ES_n and $ES_n \times_{S_n}$ Inj$_\bullet([n])$ with nice topological interpretations.

Exercise 15. (ES_n and $ES_n \times_{S_n}$ Inj$_\bullet([n])$).

(a) Let $F_n(\mathbb{R}^\infty)$ denote the space

$$F_n(\mathbb{R}^\infty) = \{(z_1, z_2, \ldots, z_n) \in (\mathbb{R}^\infty)^n \mid z_i \neq z_j \text{ for all } i \neq j\}$$

topologized as a subspace of $(\mathbb{R}^\infty)^n$. Show that $F_n(\mathbb{R}^\infty)$ has a free S_n action.

(b) Show that $F_n(\mathbb{R}^\infty)$ is contractible. *Hint:* Show that there is a fibration

$$\mathbb{R}^\infty \setminus \{(n-1) \text{ points}\} \longrightarrow F_n(\mathbb{R}^\infty) \longrightarrow F_{n-1}(\mathbb{R}^\infty)$$

and consider the long exact sequence on homotopy groups.

(c) Show that $ES_n \times_{S_n} \mathrm{Inj}_\bullet([n])$ can be interpreted as follows: a p-simplex is a choice of n distinct points in $\mathbb{R}^\infty$, where $(p+1)$ of the points are labelled by distinct letters in $[n]$. See Figure 14.

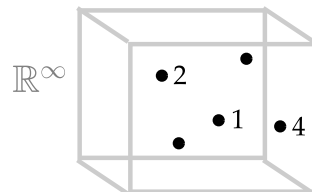


Figure 14: A point in $ES_5 \times_{S_5} \mathrm{Inj}_2([5])$

4 Suggested further reading

- Friedman, "An elementary illustrated introduction to simplicial sets." [Fr]
- Ebert–Randal-Williams, "Semisimplicial spaces." [ERW]
  This reference requires some comfort with the language of homotopy theory.
- Bestvina, "Homological stability of $\mathrm{Aut}(F_n)$ revisited." [B]
- Randal-Williams, "Homological stability for unordered configuration spaces." [RW]

References

[B] Bestvina, Mladen. "Homological stability of $\mathrm{Aut}(F_n)$ revisited." Hyperbolic Geometry and Geometric Group Theory. Mathematical Society of Japan, 2017.

[ERW] Ebert, Johannes, and Oscar Randal-Williams. "Semisimplicial spaces." Algebraic & Geometric Topology 19.4 (2019): 2099-2150.

[Fa] Farmer, Frank D. "Cellular homology for posets." Math. Japon 23.6 (1978): 79.

[Fr] Friedman, G. "An elementary illustrated introduction to simplicial sets. 2008." arXiv preprint arXiv:0809.4221.

[RW] Randal-Williams, Oscar. "Homological stability for unordered configuration spaces." The Quarterly Journal of Mathematics 64.1 (2013): 303-326.

Comments / corrections welcome!
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