The Correspondence Between String Motions and Automorphisms of the Free Group

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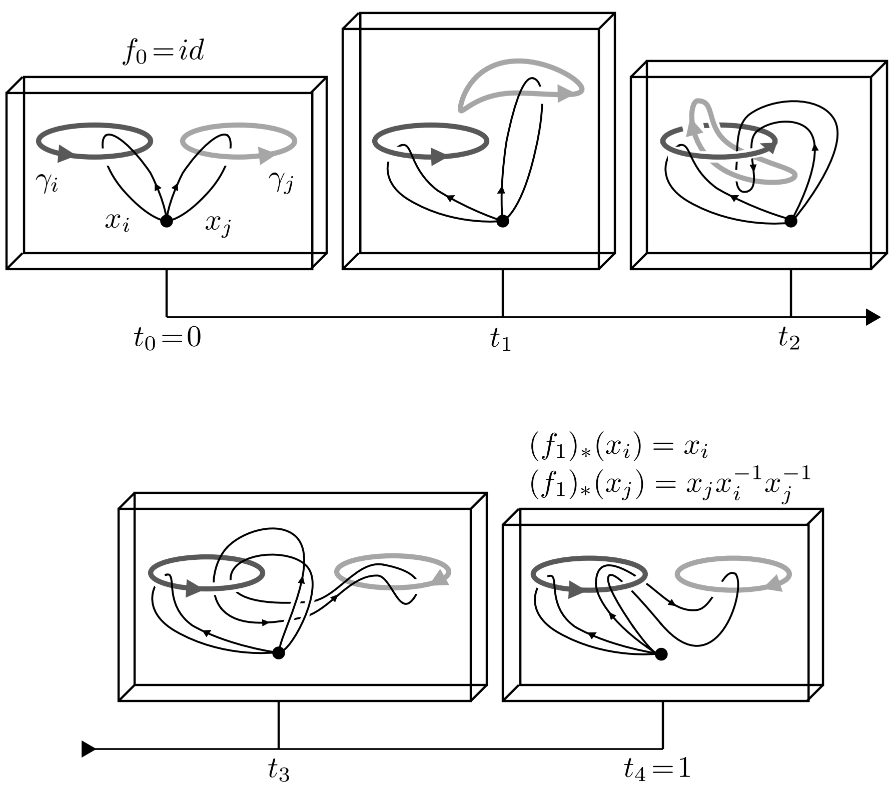
Let $C_n = \gamma_1 \cup \gamma_2 \cup \cdots \cup \gamma_n$ be the union of n disjoint, unlinked, unknotted circles in $\mathbb{R}^3$. A *motion* of C_n is a path of diffeomorphisms $f_t \in \text{Diff}(\mathbb{R}^3)$ such that f_0 is the identity and f_1 stabilizes C_n as a set, modulo the following equivalence relation: Motions $f_{t,0}$ and $f_{t,1}$ are equivalent if there is an isotopy $f_{t,s}$ such that $f_{0,s}$ and $f_{1,s}$ stabilize C_n. The product of two motions f_t and g_t is given by

$$(g \cdot f)_t = \begin{cases} f_{2t} & 0 \leq t \leq \frac{1}{2}; \\ g_{2t-1} & \frac{1}{2} \leq t \leq 1. \end{cases}$$

With this product, the set of homotopy classes of motions forms the *string motion group* Σ_n. The group identity is called the *stationary motion*.

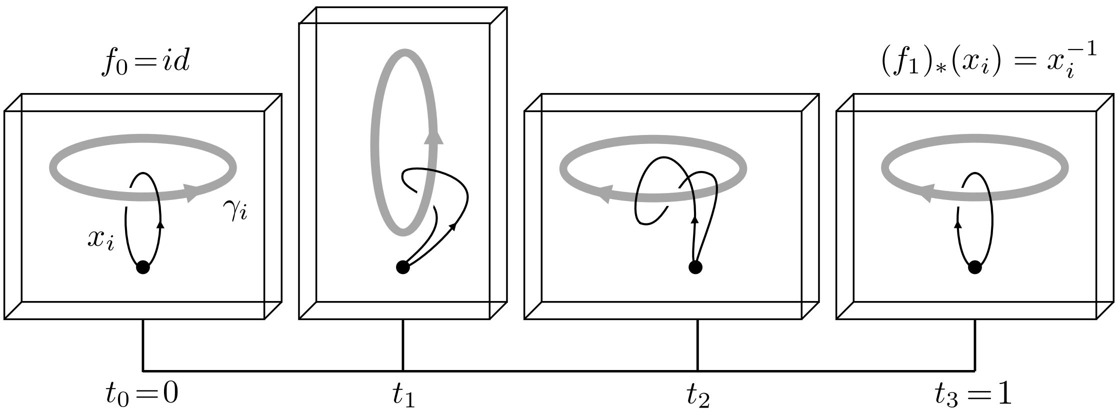
The fundamental group $\pi_1(\mathbb{R}^3 \setminus C_n)$ is the free group F_n on n generators, and we let x_i denote the generator that is linked with γ_i and unlinked with the other circles. We can embed Σ_n into the automorphism group $\text{Aut}(F_n)$ by identifying each motion $\{f_t\}$ with the map $(f_1)_* : \pi_1(\mathbb{R}^3 \setminus C_n) \to \pi_1(\mathbb{R}^3 \setminus C_n)$. An example of a string motion f_t, and the corresponding automorphism, is shown in the following picture:



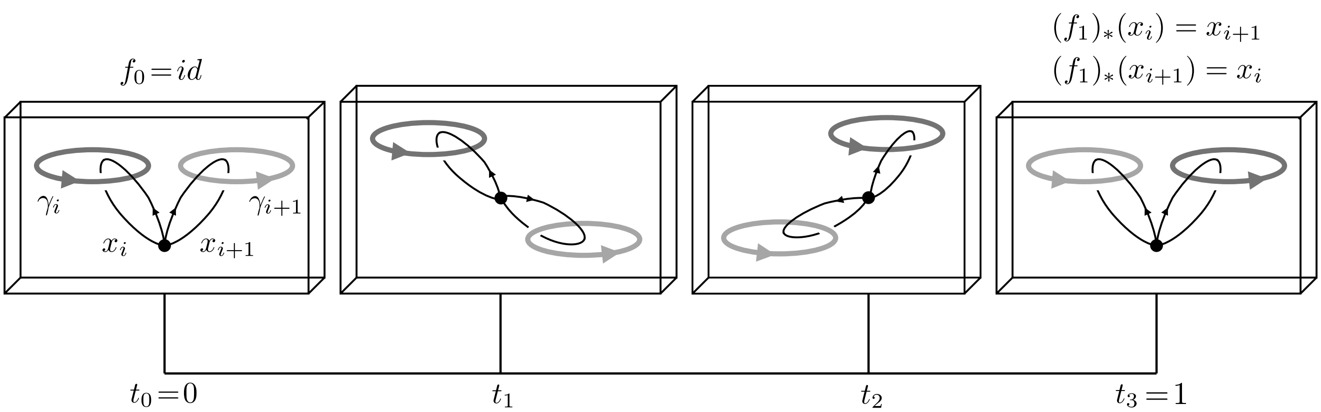
The image of Σ_n in $\mathrm{Aut}(F_n)$ is the subgroup of generated by elements $\alpha_{i,j}$, ρ_i, and τ_i, the automorphisms determined as follows:

$$\rho_i = \begin{cases} x_i \mapsto x_i^{-1} \\ x_k \mapsto x_k \quad (k \neq i) \end{cases} \qquad \tau_i = \begin{cases} x_i \mapsto x_{i+1} \\ x_{i+1} \mapsto x_i \\ x_k \mapsto x_k \quad (k \neq i,\, i+1) \end{cases} \qquad \alpha_{i,j} = \begin{cases} x_i \mapsto x_j x_i x_j^{-1} \\ x_k \mapsto x_k \quad (k \neq i) \end{cases} .$$

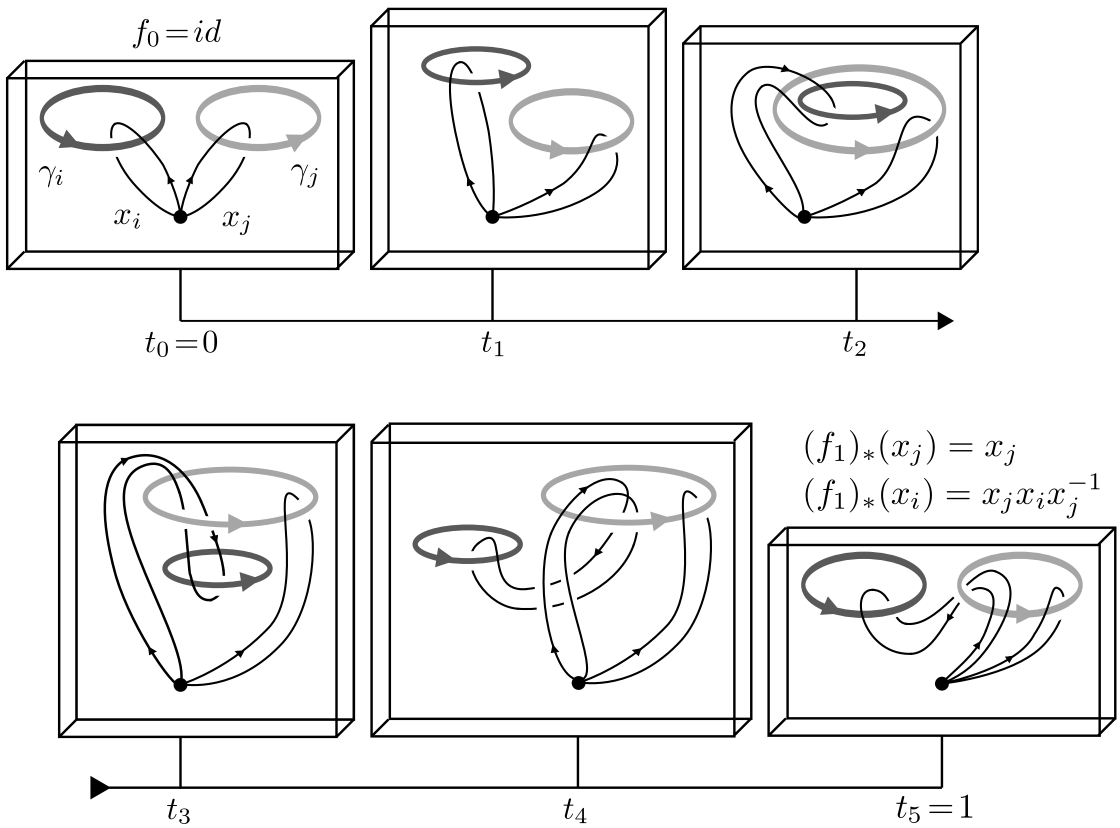
The automorphism ρ_i is induced by a string motion that reverses the orientation of the circle γ_i:



The automorphism τ_i is induced by a string motion that permutes the circles γ_i with γ_{i+1} while preserving their orientations:



The automorphism $\alpha_{i,j}$, $i \neq j$, corresponds to a string motion where the i^{th} circle γ_i passes through the j^{th} circle γ_j, and returns to its original position and orientation:



These automorphisms $\alpha_{i,j}, i \neq j$, generate the *pure string motion group*.