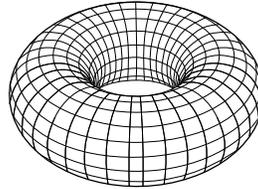


The Definition of a Manifold and First Examples

In brief, a (real) n -dimensional manifold is a topological space \mathcal{M} for which every point $x \in \mathcal{M}$ has a neighbourhood homeomorphic to Euclidean space \mathbb{R}^n .

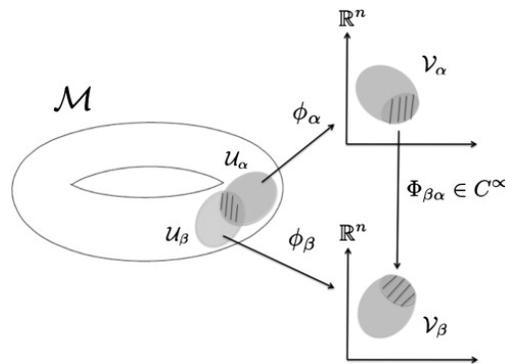


Definition 1. (Coordinate system, Chart, Parameterization) Let \mathcal{M} be a topological space and $\mathcal{U} \subseteq \mathcal{M}$ an open set. Let $\mathcal{V} \subseteq \mathbb{R}^n$ be open. A homeomorphism $\phi : \mathcal{U} \rightarrow \mathcal{V}$, $\phi(u) = (x_1(u), \dots, x_n(u))$ is called a *coordinate system* on \mathcal{U} , and the functions x_1, \dots, x_n the *coordinate functions*. The pair (\mathcal{U}, ϕ) is called a *chart* on \mathcal{M} . The inverse map ϕ^{-1} is a *parameterization* of \mathcal{U} .

Definition 2. (Atlas, Transition maps) An *atlas* on \mathcal{M} is a collection of charts $\{\mathcal{U}_\alpha, \phi_\alpha\}$ such that \mathcal{U}_α cover \mathcal{M} . The homeomorphisms $\phi_\beta \phi_\alpha^{-1} : \phi_\alpha(\mathcal{U}_\alpha \cap \mathcal{U}_\beta) \rightarrow \phi_\beta(\mathcal{U}_\alpha \cap \mathcal{U}_\beta)$ are the *transition maps* or *coordinate transformations*.

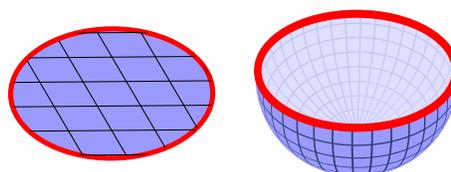
Recall that a topological space is *second countable* if the topology has a countable base, and *Hausdorff* if distinct points can be separated by neighbourhoods.

Definition 3. (Topological manifold, Smooth manifold) A second countable, Hausdorff topological space \mathcal{M} is an n -dimensional *topological manifold* if it admits an atlas $\{\mathcal{U}_\alpha, \phi_\alpha\}$, $\phi_\alpha : \mathcal{U}_\alpha \rightarrow \mathbb{R}^n$, $n \in \mathbb{N}$. It is a *smooth manifold* if all transition maps are C^∞ diffeomorphisms, that is, all partial derivatives exist and are continuous.



Two smooth atlases are *equivalent* if their union is a smooth atlas. In general, a *smooth structure* on \mathcal{M} may be defined as an equivalence class of smooth atlases, or as a maximal smooth atlas.

Definition 4. (Manifold with boundary, Boundary, Interior) We define a n -dimensional *manifold with boundary* \mathcal{M} as above, but now allow the image of each chart to be an open subset of Euclidean space \mathbb{R}^n or an open subset of the upper half-space $\mathbb{R}_+^n := \{(x_1, \dots, x_n) \mid x_n \geq 0\}$. The preimages of points $(x_1, \dots, x_{n-1}, 0) \in \mathbb{R}_+^n$ are the *boundary* $\partial\mathcal{M}$ of \mathcal{M} , and $\mathcal{M} - \partial\mathcal{M}$ is the *interior* of \mathcal{M} .



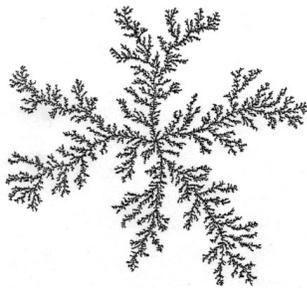
A manifold with boundary is *smooth* if the transition maps are smooth. Recall that, given an arbitrary subset $X \subseteq \mathbb{R}^m$, a function $f : X \rightarrow \mathbb{R}^n$ is called smooth if every point in X has some neighbourhood where f can be extended to a smooth function.

Definition 5. A function $f : \mathcal{M} \rightarrow \mathcal{N}$ is a *map of topological manifolds* if f is continuous. It is a *smooth map of smooth manifolds* \mathcal{M}, \mathcal{N} if for any smooth charts (\mathcal{U}, ϕ) of \mathcal{M} and (\mathcal{V}, ψ) of \mathcal{N} , the function

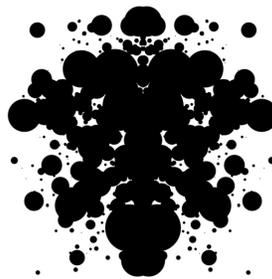
$$\psi \circ f \circ \phi^{-1} : \phi(\mathcal{U} \cap f^{-1}(\mathcal{V})) \rightarrow \psi(\mathcal{V})$$

is a C^∞ diffeomorphism.

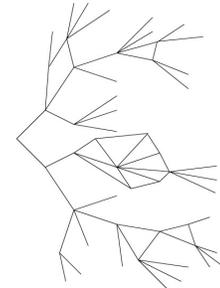
Exercise # 1. (Recognizing Manifolds) Which of the following have a manifold structure (possibly with boundary)?



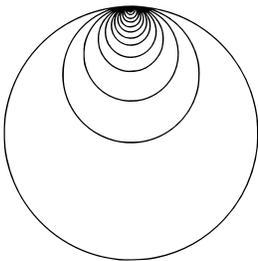
(a) Arbitrary subset $X \subseteq \mathbb{R}^n$



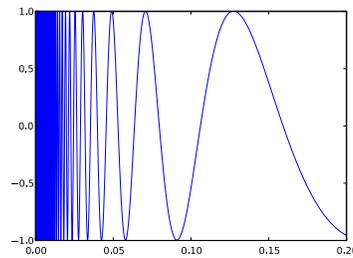
(b) Arbitrary open subset $U \subseteq \mathbb{R}^n$



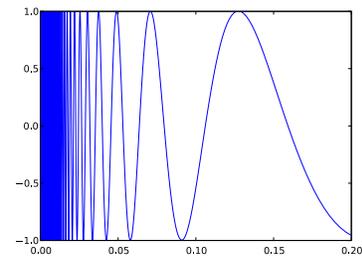
(c) Graph



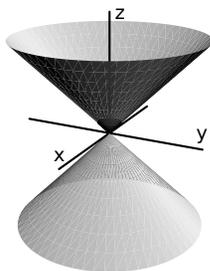
(d) Hawaiian Earring



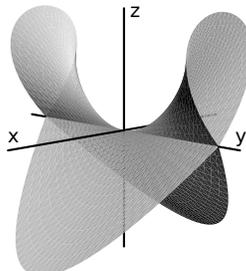
(e) $\{(x, \sin \frac{1}{x}) \mid x \in (0, 1]\}$



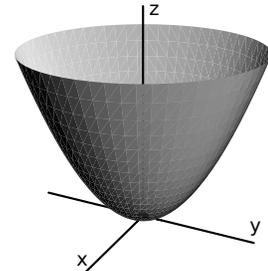
(f) $\{(x, \sin \frac{1}{x}) \mid x \in (0, 1]\} \cup (\{0\} \times [0, 1])$



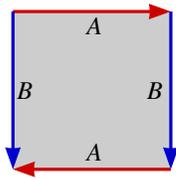
(g) Solution to $x^2 + y^2 = z^2$



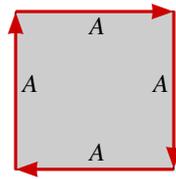
(h) Solution to $x^2 + z^3 = y^2 z^2$



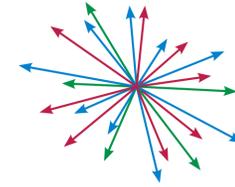
(i) Solution to $z = x^2 + y^2$



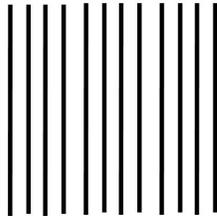
(j) Quotient Space $ABAB^{-1}$



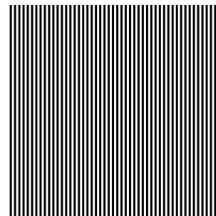
(k) Quotient Space $AAAA$



(l) Infinite Coordinate Space $\bigoplus_{\mathbb{N}} \mathbb{R}$



(m) Disjoint union $\coprod_{\mathbb{N}} [0, 1]$ of discrete copies of $[0, 1]$



(n) Disjoint union $\coprod_{\text{continuum}} [0, 1]$ of discrete copies of $[0, 1]$



(o) Infinite Real Hilbert Space

It is a difficult fact that not every topological manifold admits a smooth structure. Moreover, a topological manifold may have multiple nondiffeomorphic smooth structures. For example, there are uncountably many distinct smooth structures on \mathbb{R}^4 .

Exercise # 2. (Atlases on the circle) Define the 1-sphere S^1 to be the unit circle in \mathbb{R}^2 . Put atlases on S^1 using charts defined by:

1. the angle of rotation from a fixed point,
2. the slope of the secant line from a fixed point,
3. the projections to the x and y axes.

In each case, compute the transition functions.

Exercise # 3. (Topological vs. Smooth) Give an example of a topological space \mathcal{M} and an atlas on \mathcal{M} that makes \mathcal{M} a topological, but not smooth, manifold.

Exercise # 4. (Products of manifolds) Let \mathcal{M} be a manifold with atlas (U_α, ϕ_α) , and \mathcal{N} a manifold with atlas (V_β, ψ_β) , and assume at least one of $\partial\mathcal{M}$ and $\partial\mathcal{N}$ is empty. Describe a manifold structure on the Cartesian product $\mathcal{M} \times \mathcal{N}$.

Exercise # 5. (Manifold boundaries) Let \mathcal{M} be an n -dimensional manifold with nonempty boundary $\partial\mathcal{M}$. Show that $\partial\mathcal{M}$ is an $(n - 1)$ -dimensional manifold with empty boundary.

Exercise # 6. (Atlases on spheres) Prove that any atlas on S^1 must include at least two charts. Do the same for the 2-sphere S^2 .

Exercise # 7. (Centering charts) Given a (topological or smooth) manifold \mathcal{M} , and any $x \in \mathcal{M}$, show that there is some chart (U, ϕ) with $x \in U$ centered at x in the sense that $\phi(x) = 0$. Show similarly that there is some parameterization ψ of U such that $\psi(0) = x$.

Definition 6. (Real and complex projective spaces) The *projectivization* of a vector space V is the space of 1-dimensional subspaces of V . Real and complex projective spaces are defined:

$$\mathbb{R}P^n := (\mathbb{R}^{n+1} - \{0\}) / (x_0, \dots, x_n) \sim (cx_0, \dots, cx_n) \quad \text{for } c \in \mathbb{R}$$

$$\mathbb{C}P^n := (\mathbb{C}^{n+1} - \{0\}) / (x_0, \dots, x_n) \sim (cx_0, \dots, cx_n) \quad \text{for } c \in \mathbb{C}$$

Points in $\mathbb{R}P^n$ and $\mathbb{C}P^n$ are often written in *homogeneous coordinates*, where $[x_0 : x_1 : \dots : x_n]$ denotes the equivalence class of the point (x_0, x_1, \dots, x_n) .

Exercise # 8. Let $\mathcal{U}_i = \{[x_0 : x_1 : \dots : x_n] \in \mathbb{R}P^n \mid x_i \neq 0\}$. Use the sets \mathcal{U}_i to define a manifold structure on $\mathbb{R}P^n$.

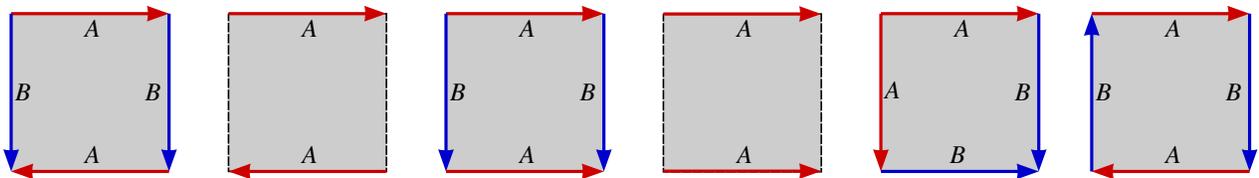
Exercise # 9. Show that $\mathbb{R}P^n$ can also be defined as the quotient of n -sphere S^n by the antipodal map, identifying a point (x_0, \dots, x_n) with $(-x_0, \dots, -x_n)$.

Exercise # 10. Show that $\mathbb{R}P^n$ decomposes as the disjoint union $\mathbb{R}P^n = \mathbb{R}^n \sqcup \mathbb{R}P^{n-1}$, and hence inductively

$$\mathbb{R}P^n = \mathbb{R}^n \sqcup \mathbb{R}^{n-1} \sqcup \dots \sqcup \mathbb{R}^1 \sqcup \text{point}.$$

These subspaces are called the *point at infinity*, the *line at infinity*, etc.

Exercise # 11. Identify among the following quotient spaces: a cylinder, a Möbius band, a sphere, a torus, real projective space, and a Klein bottle.



Exercise # 12. (The cylinder as a quotient) Define the cylinder C to be the subset of \mathbb{R}^3

$$C = \{(\cos \theta, \sin \theta, z) \mid 0 \leq \theta < 2\pi, 0 \leq z \leq 1\}.$$

Give a rigorous proof that C is homeomorphic to the unit square $R = [0, 1] \times [0, 1]$ in \mathbb{R}^2 modulo the identification $(0, x) \sim (1, x)$ for all $x \in [0, 1]$.

Exercise # 13. ($T^2 \cong \mathbb{R}^2/\mathbb{Z}^2$) Show that the quotient of the plane \mathbb{R}^2 by the action of \mathbb{Z}^2 is the torus $T^2 = S^1 \times S^1$,

$$T^2 = \mathbb{R}^2 / (x, y) \sim ((x, y) + (m, n)) \quad \text{for } (m, n) \in \mathbb{Z}^2.$$

Exercise # 14. (Challenge) (Classification of 1-manifolds) Prove that any smooth, connected 1-manifold is diffeomorphic to the circle S^1 or to an interval of \mathbb{R} .

Exercise # 15. (Challenge) (Topological groups) Show that the following groups have the structure of a manifold. Compute their dimension, find the number of connected components, and determine whether they are compact.

1. Real $n \times n$ matrices $M_n(\mathbb{R})$

2. Rigid motions of Euclidean space $E_n(\mathbb{R})$
3. $m \times n$ matrices of maximal rank
4. General linear group $GL_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) \mid \det(A) \neq 0\}$
5. Special linear group $SL_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) \mid \det(A) = 1\}$
6. Orthogonal group $O_n(\mathbb{R}) = \{A \in GL_n(\mathbb{R}) \mid \text{transpose}(A) = A^{-1}\}$
7. Special orthogonal group $SO_n(\mathbb{R}) = O_n(\mathbb{R}) \cap SL_n(\mathbb{R})$

Exercise # 16. (Challenge) (The real Grassmannian) The projective space of a vector space V is a special case of the Grassmannian $G(r, V)$, the space of r -planes through the origin. Show that, as a set,

$$G(r, \mathbb{R}^n) \cong O(n)/(O(r) \times O(n-r)).$$

Argue that this identification gives $G(r, \mathbb{R}^n)$ the structure of a smooth compact manifold, and compute its dimension.

Multivariate Calculus

Definition 7. (Derivative) Given a function $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$, the derivative of f at x is the linear map

$$df_x : \mathbb{R}^n \rightarrow \mathbb{R}^m$$

$$y \mapsto \lim_{t \rightarrow 0} \frac{f(\mathbf{x} + t\mathbf{y}) - f(\mathbf{x})}{t} \quad \text{The directional derivative of } f \text{ at } x \text{ in the } y \text{ direction}$$

Write f in coordinates $f(x) = (f_1(x), \dots, f_m(x))$. If all first partial derivatives exist and are continuous in a neighbourhood of x , then df_x exists and is given by the *Jacobian*, the $m \times n$ matrix

$$\begin{bmatrix} \frac{df_1}{dx_1}(x) & \cdots & \frac{df_1}{dx_n}(x) \\ \vdots & & \vdots \\ \frac{df_m}{dx_1}(x) & \cdots & \frac{df_m}{dx_n}(x) \end{bmatrix}$$

The chain rule asserts that given smooth maps f, g

$$\begin{array}{ccccc} & & g \circ f & & \\ & \curvearrowright & & \curvearrowleft & \\ \mathcal{U} & \xrightarrow{f} & \mathcal{V} & \xrightarrow{g} & \mathcal{W} \\ \supset & & \supset & & \supset \\ \mathbb{R}^n & & \mathbb{R}^m & & \mathbb{R}^\ell \end{array}$$

then for each $x \in \mathcal{U}$, the following diagram commutes:

$$\begin{array}{ccccc} & & d(g \circ f)_x & & \\ & \curvearrowright & \text{---} & \curvearrowleft & \\ \mathbb{R}^n & \xrightarrow{df_x} & \mathbb{R}^m & \xrightarrow{dg_{f(x)}} & \mathbb{R}^\ell \end{array}$$

Exercise # 17. Let f and g be functions from $\mathbb{R}^n \rightarrow \mathbb{R}$ with derivatives df_x and dg_x at x . Give formulas for the derivative of their pointwise sum and product, $f + g$ and fg . Conclude that the space of r -times differentiable functions $\mathbb{R}^n \rightarrow \mathbb{R}$ is closed under addition and multiplication. Repeat the exercise for functions $\mathbb{R}^n \rightarrow \mathbb{R}^m$ under sum and scalar product.

Exercise # 18. Let f be an r -times differentiable function $\mathbb{R}^n \rightarrow \mathbb{R}$. Under what circumstances is the pointwise reciprocal $\frac{1}{f}$ an r -times differentiable function?

Theorem 8. (Taylor’s Theorem) Given an n -tuple $\alpha = (\alpha_1, \dots, \alpha_n)$, let

$$|\alpha| = \alpha_1 + \dots + \alpha_n \quad \alpha! = \alpha_1! \dots \alpha_n! \quad x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n} \quad D^\alpha f = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}}$$

and let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a $(r + 1)$ times continuously differentiable function in a ball B of y . Then for $x \in B$, f has a Taylor expansion

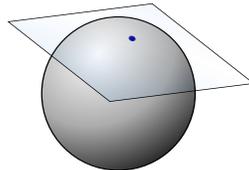
$$f(x) = \sum_{|\alpha|=0}^r \frac{D^\alpha f(y)}{\alpha!} (x - y)^\alpha + \sum_{|\beta|=r+1} R_\beta(x) (x - y)^\beta$$

where the remainder

$$R_\beta(x) = \frac{|\beta|}{\beta!} \int_0^1 (1 - t)^{|\beta|-1} D^\beta f(y + t(x - y)) dt$$

Tangent Spaces and Derivatives

Definition 9. (Tangent Space $T_x\mathcal{M}$, Derivatives) Suppose that \mathcal{M} is a smooth m -dimensional submanifold of some Euclidean space \mathbb{R}^N . (We will see in Theorem 18 that every manifold can be realized this way). Let $\phi : \mathcal{U} \rightarrow \mathcal{M}$ be a local parameterization around some point $x \in \mathcal{M}$ with $\phi(0) = x$. We define the *tangent space* $T_x\mathcal{M}$ to be the image of the map $d\phi_0 : \mathbb{R}^m \rightarrow \mathbb{R}^N$. Note that $T_x\mathcal{M}$ is an m -dimensional subspace of \mathbb{R}^N ; its translate $x + T_x\mathcal{M}$ is the best flat approximation to ϕ at x .



Given a smooth map of manifolds $f : \mathcal{M} \rightarrow \mathcal{N}$, and local parameterizations $\phi : \mathcal{U} \rightarrow \mathcal{M}$, $\phi(0) = x \in \mathcal{M}$ and $\psi : \mathcal{V} \rightarrow \mathcal{N}$, $\psi(0) = f(x) \in \mathcal{N}$. Let h be the map $h = \psi^{-1} \circ f \circ \phi : \mathcal{U} \rightarrow \mathcal{V}$. We can define the *differential of f at x* by

$$\begin{aligned} df_x &: T_x\mathcal{M} \rightarrow T_{f(x)}\mathcal{N} \\ df_x &= d\psi_0 \circ dh_0 \circ d\phi_0^{-1}. \end{aligned}$$

The spaces $T_x(\mathcal{M})$, $T_{f(x)}(\mathcal{N})$, and the differential df_x are independent of choice of local parameterizations ϕ and ψ .

Exercise # 19. (Tangent spaces to products) Given smooth manifolds \mathcal{M} and \mathcal{N} , show that

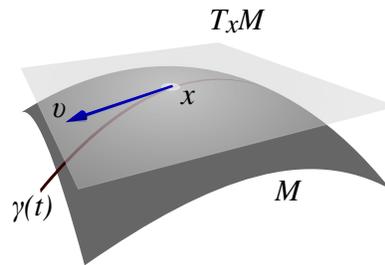
$$T_{(x,y)}(\mathcal{M} \times \mathcal{N}) \cong T_x\mathcal{M} \times T_y\mathcal{N}.$$

Exercise # 20. (Tangent spaces to vector spaces) Show that if V is a vector subspace of \mathbb{R}^N , then for $x \in V$, $T_xV = V$.

Exercise # 21. (Chain rule for manifolds) Prove that if $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ are smooth maps of manifolds,

$$d(g \circ f)_x = dg_{f(x)} \circ df_x.$$

There are alternate, intrinsic definitions of the tangent space $T_x\mathcal{M}$ that do not require \mathcal{M} to lie in \mathbb{R}^N . Given \mathcal{M} and $x \in \mathcal{M}$, we define a *curve through x* to be a smooth map $\gamma : \mathbb{R} \rightarrow \mathcal{M}$ with $\gamma(0) = x$. Choose a chart (\mathcal{U}, ϕ) , $\mathcal{U} \ni x$; we call two curves γ_1 and γ_2 through x *equivalent* if $(\phi \circ \gamma_1)'(0) = (\phi \circ \gamma_2)'(0)$. Then we can define $T_x\mathcal{M}$ as the equivalence classes of curves through x ; it is independent of choice of chart.



In this formulation, the differential df_x of a map $f : \mathcal{N} \rightarrow \mathcal{M}$ is defined as the map

$$\begin{aligned} df_x : T_x\mathcal{M} &\longrightarrow T_{f(x)}\mathcal{N} \\ [\gamma : \mathbb{R} \rightarrow \mathcal{M}] &\longmapsto [f \circ \gamma : \mathbb{R} \rightarrow \mathcal{N}] \end{aligned}$$

The Constant Rank Theorem and Consequences

The Constant Rank Theorem states that the local behaviour of a smooth map $f : \mathcal{U} \rightarrow \mathbb{R}^n$ is governed by the rank of the Jacobian.

Theorem 10. (Constant Rank Theorem) Let $\mathcal{U} \subseteq \mathbb{R}^m$ be open, and let $f : \mathcal{U} \rightarrow \mathbb{R}^n$ be a smooth map. If the Jacobian of f is of constant rank k in a neighbourhood of p , then there are local coordinate systems g at p and h at $f(p)$ such that

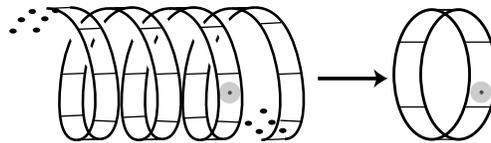
$$hfg^{-1}(x_1, \dots, x_m) = (y_1, \dots, y_k, 0, \dots, 0)$$

In other words, in appropriately chosen coordinates, f is locally the composition of the projection $\mathbb{R}^m \rightarrow \mathbb{R}^k$ and the inclusion $\mathbb{R}^k \hookrightarrow \mathbb{R}^n$.

The case of the Constant Rank Theorem with $n = k = m$ implies the Inverse Function Theorem:

Theorem 11. (Inverse Function Theorem) Suppose that \mathcal{M}, \mathcal{N} are smooth manifolds, and $f : \mathcal{M} \rightarrow \mathcal{N}$ is a smooth map whose derivative df_x at x is an isomorphism. Then f is a local diffeomorphism at x , that is, f takes some neighbourhood of x diffeomorphically onto its image.

In this case, f locally has a smooth inverse f^{-1} , with differential $d(f^{-1})_{f(x)} = (df_x)^{-1}$ at $f(x)$.

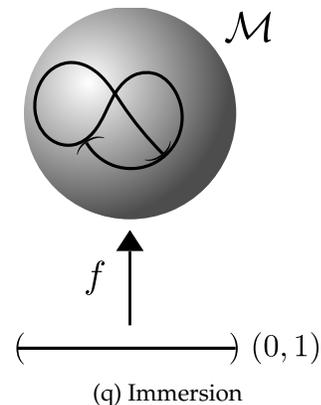
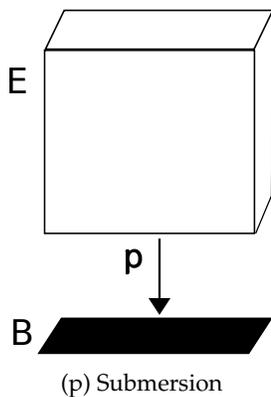


Local diffeomorphism

Exercise # 22. (Local diffeomorphism at each point vs. global diffeomorphism) Show by example that even if a map $f : X \rightarrow Y$ is a local diffeomorphism at each point in X , it may not be a global diffeomorphism.

Exercise # 23. (Local Immersion and Submersion Theorems) Suppose that \mathcal{M} is an m -dimensional manifold and \mathcal{N} is an n -dimensional manifold, $m \leq n$. Let $f : \mathcal{M} \rightarrow \mathcal{N}$ be a smooth map such that df_x is injective at some point x . Show that the Constant Rank Theorem implies that, in appropriately chosen local coordinates, f is the natural inclusion $\mathbb{R}^m \hookrightarrow \mathbb{R}^n$ in a neighbourhood of x . Similarly, show that if a map $g : \mathcal{N} \rightarrow \mathcal{M}$ has a surjective differential dg_y at some point y , then there are local coordinates such that g is the projection $\mathbb{R}^n \rightarrow \mathbb{R}^m$ in some neighbourhood of y .

Definition 12. (Submersion; Immersion) A map $f : \mathcal{M} \rightarrow \mathcal{N}$ of smooth manifolds is a *submersion* if its derivative df_x is surjective at each x . The map is an *immersion* if the derivative is injective at every x . The image of an immersion is sometimes called an *immersed submanifold*, though it may not have a manifold structure.



Definition 13. (Embedding, Submanifold) A function $f : \mathcal{M} \rightarrow \mathcal{N}$ of topological manifolds is a *topological embedding* if it is a homeomorphism onto its image. If \mathcal{M} and \mathcal{N} are smooth manifolds, then f is a *smooth embedding* if it is an immersion and a topological embedding. Some references additionally require embeddings to be *proper* maps, that is, preimages of compact sets must be compact. With either definition, the image of an embedding is a (*regular*) *submanifold*; the topology on \mathcal{M} coincides with the subspace topology on $f(\mathcal{M})$.

Exercise # 24. (Image of an immersion) Give an example of an immersion $i : \mathcal{M} \rightarrow \mathcal{N}$ such that $i(\mathcal{M})$ is not a submanifold of \mathcal{N} .

Exercise # 25. (Injective immersion vs. Embedding) Give an example of a map of smooth manifolds $f : \mathcal{M} \rightarrow \mathcal{N}$ that is injective and an immersion, but not an embedding. Is an injective local diffeomorphism necessarily a global diffeomorphism?

Exercise # 26. (Two definitions of embedding) Give an example of a map of smooth manifolds $f : \mathcal{M} \rightarrow \mathcal{N}$ that is an immersion and a homeomorphism onto its image, but not proper. Would you call f an embedding?

Exercise # 27. (Proper injective immersions are embeddings) Prove that a proper injective immersion of smooth manifolds $f : \mathcal{M} \rightarrow \mathcal{N}$ is an embedding. Conclude that if \mathcal{M} is compact then the embeddings $f : \mathcal{M} \rightarrow \mathcal{N}$ are precisely the injective immersions.

Exercise # 28. (Paths in the torus) In Exercise # 13, we realized the torus as the orbit space $q : \mathbb{R}^2/\mathbb{Z}^2 \cong T^2$. Consider the map $f : \mathbb{R} \rightarrow T^2$ defined by the image of the graph $y = mx$

$$\begin{aligned} f : \mathbb{R} &\longrightarrow \mathbb{R}^2 &\longrightarrow \mathbb{R}^2/\mathbb{Z}^2 \\ x &\longmapsto (x, mx) &\longmapsto q(x, mx) \end{aligned}$$

Under what circumstances is this f an immersion? When is it an embedding? When is its image a submanifold?

Exercise # 29. The solution sets to the following polynomials are pictured in Exercise # 1: $x^2 + y^2 = z^2$, $x^2 + z^3 = y^2z^2$, and $z = x^2 + y^2$. Use these formulas to determine which of the solutions sets are embedded submanifolds of \mathbb{R}^3 .

Critical Values and Sard's Theorem

Definition 14. (Critical values, Regular values) Given a smooth map of manifolds $f : \mathcal{M} \rightarrow \mathcal{N}$, the set of *critical values* of f is defined as

$$\{f(p) \in \mathcal{N} \mid \text{rank}(df_p) < \dim(\mathcal{N})\}.$$

The *regular values* of f are the complement of the critical values in \mathcal{N} , that is,

$$\{q \in \mathcal{N} \mid \text{rank}(df_p) = \dim(\mathcal{N}) \text{ at every preimage } p \in f^{-1}(q)\} \cup \{q \in \mathcal{N} \mid q \notin f(\mathcal{M})\}.$$

Theorem 15. (Sard's Theorem) *The set of critical values of a smooth map of manifolds $f : \mathcal{M} \rightarrow \mathcal{N}$ has measure 0 in \mathcal{N} , in the sense that its preimage in Euclidean space under any local parameterization has measure 0.*

Exercise # 30. (Critical points, Regular points) Given a smooth map $f : \mathcal{M} \rightarrow \mathcal{N}$, a point p in \mathcal{M} is a *critical point* if $\text{rank}(df_p) < \dim(\mathcal{N})$, otherwise it is a *regular point*. Note that regular and critical points lie in the domain of f , while regular and critical values lie in the codomain. Show by example that the set of critical points may have positive measure in \mathcal{M} .

Exercise # 31. (Preimage Theorem) Let \mathcal{M} and \mathcal{N} be smooth manifolds, and $f : \mathcal{M} \rightarrow \mathcal{N}$. Suppose $q \in f(\mathcal{M}) \subseteq \mathcal{N}$ is a regular value of f . Use the Constant Rank Theorem show that its preimage $f^{-1}(q)$ is a submanifold of \mathcal{M} , with dimension $\dim(\mathcal{M}) - \dim(\mathcal{N})$.

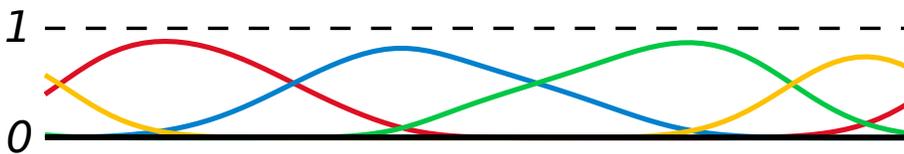
Exercise # 32. (Measure of submanifolds) Prove that submanifolds of positive codimension have measure 0.

Exercise # 33. (Dense critical values) Show by example that the critical values of a map $f : \mathbb{R} \rightarrow \mathbb{R}$ may be dense.

Partitions of Unity

An essential tool for developing the theory of real manifolds is the existence of partitions of unity:

Definition 16. (Partition of unity) Given a smooth manifold \mathcal{M} , a *partition of unity* on \mathcal{M} is a set of smooth functions $\{\rho_i\}_I$, $\rho_i : \mathcal{M} \rightarrow [0, 1]$, so that each $m \in \mathcal{M}$ has a neighbourhood where only finitely many ρ_i are nonzero, and $\sum_I \rho_i(m) = 1$.



Theorem 17. (Partitions of unity exist) Given a manifold \mathcal{M} and an open cover $\{\mathcal{U}_\alpha\}_A$ of \mathcal{M} , there exists a partition of unity $\{\rho_\alpha\}_A$ subordinate to \mathcal{U} in the sense that the support of ρ_α is contained in \mathcal{U}_α . Alternatively, there exists some partition of unity $\{\rho_i\}_I$ so that each ρ_i has compact support contained in some \mathcal{U}_α .

Partitions of unity allow us to extend certain structures or operations defined in individual coordinate systems to the whole manifolds. Examples are integration, bundle sections (such as vector fields, Riemannian metrics, ...).

Exercise # 34. (Existence of proper maps) Prove that any manifold \mathcal{M} admits a proper map $\rho : \mathcal{M} \rightarrow \mathbb{R}$.

Whitney Embedding Theorem

Theorem 18. (Whitney Embedding Theorem) For $m > 0$, any smooth m -dimensional manifold admits a smooth proper embedding into \mathbb{R}^{2m} .

This dimension $2m$ is a sharp lower bound: $\mathbb{R}P^m$ cannot embed in \mathbb{R}^{2m-1} whenever m is a power of 2.

Exercise # 35. (Challenge) (Weak Whitney Immersion Theorem) Prove a weaker version of Theorem 18: any m -dimensional manifold admits an injective immersion into \mathbb{R}^{2m+1} .

Exercise # 36. (Challenge) (Weak Whitney Embedding Theorem) Use Exercises # 34 and # 35 to prove that every m -manifold properly embeds in \mathbb{R}^{2m+1} .

Further Reading

Munkres, *Topology: A First Course*.

Spivak, *Calculus on Manifolds*.

Guillemin–Pollack, *Differential Topology*.

Lee, *Introduction to Smooth Manifolds*.

Milnor, *Topology from the Differential Viewpoint*.

Madsen–Tornehave, *From Calculus to Cohomology*.

Sources for graphics

Wireframe torus: Yassine Mrabet, http://commons.wikimedia.org/wiki/File:Simple_Torus.svg

Transition functions: modified from Andrew Lewis, *Lagrangian Mechanics, Dynamics, and Control*

Disk and hemisphere with boundary: Yassine Mrabet,

<http://commons.wikimedia.org/wiki/File:SurfacesWithAndWithoutBoundary.svg>

Hawaiian earring: http://commons.wikimedia.org/wiki/File:Hawaiian_earrings.svg

Algebraic graphs: Cox, Little, O'Shea, *Ideals, Varieties, and Algorithms*

Surfaces via edge identifications: modified from Ilmari Karonen,

<http://commons.wikimedia.org/wiki/File:KleinBottleAsSquare.svg>

Tangent plane to sphere: from Alex Wright,

http://commons.wikimedia.org/wiki/File:Image_Tangent-plane.svg

Tangent vector: <http://commons.wikimedia.org/wiki/File:Tangentialvektor.svg>

Local diffeomorphism: made in Inkscape for this handout

Submersion: modified from http://en.wikipedia.org/wiki/File:Bundle_section.svg

Immersion: made in Inkscape for this handout

Partition of unity graphs: Oleg Alexandrov,

http://commons.wikimedia.org/wiki/File:Partition_of_unity_illustration.svg