

# FI-hyperhomology and ordered configuration spaces

Jeremy Miller\* and Jennifer C. H. Wilson

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## Abstract

Using a result of Gan–Li on FI-hyperhomology and a semi-simplicial resolution of configuration spaces due to Randal-Williams, we establish an improved representation stability stable range for configuration spaces of distinct ordered points in a manifold. Our bounds on generation degree improve the best known stability slope by a factor of  $5/2$  in the most general case. We adapt this result of Gan–Li to apply beyond stability arguments involving highly-connected simplicial complexes, and our methods suggest that their result may be widely applicable to improving most stability ranges for FI-modules in the current representation stability literature.

## 1 Introduction

In this paper, we prove a representation stability result for ordered configuration spaces of points in a manifold. This result improves upon previous results of Church [Chu12], Church–Ellenberg–Farb [CEF15], Church–Ellenberg–Farb–Nagpal [CEFN14], Miller–Wilson [MW], Church–Miller–Nagpal–Reinhold [CMNR18] and Bahran [Bah]: it establishes the best currently known stable ranges for the cohomology of ordered configuration spaces of manifolds with integral coefficients.

Let  $M$  be a connected manifold of dimension at least 2. The family of  $k$ -point ordered configuration spaces of  $M$  are not cohomologically stable in  $k$ ; however, their cohomology groups do stabilize with respect to their natural  $S_k$ -actions. The standard way to define and quantify this stability is through bounds on the *generation degree* and *presentation degree* of the corresponding FI-*module* structure on the sequences of cohomology groups, which we recall below. These are equivariant analogues of the classical homological stability stable ranges in which the stabilization map is surjective or, is an isomorphism, respectively.

Let FI denote the category of finite sets and injective maps. An FI-*module* is a functor from FI to the category of abelian groups, an FI-*chain complex* is a functor from FI to chain-complexes over  $\mathbb{Z}$ , and so forth. Given an FI-module  $\mathcal{V}$ , we denote its value on a finite set  $S$  by  $\mathcal{V}_S$ . Following the notation of Church–Ellenberg–Farb [CEF15], we let

$$H_0^{\text{FI}} : \text{FI-Mod} \longrightarrow \text{FI-Mod}$$

be the functor given by the formula

$$H_0^{\text{FI}}(\mathcal{V})_S := \text{coker} \left( \bigoplus_{T \subsetneq S} \mathcal{V}_T \rightarrow \mathcal{V}_S \right).$$

Here the maps  $\mathcal{V}_T \rightarrow \mathcal{V}_S$  are induced by the FI-module structure. Let  $H_i^{\text{FI}}$  denote the  $i$ th left-derived functor of  $H_0^{\text{FI}}$ . These functors are known as the FI-*homology* functors; see Church–Ellenberg [CE17]. We say that an FI-module  $\mathcal{V}$  has *generation degree*  $\leq g$  if

$$H_0^{\text{FI}}(\mathcal{V})_S \cong 0 \quad \text{for } |S| > g.$$

We say that  $\mathcal{V}$  has *presentation degree*  $\leq r$  if

$$H_0^{\text{FI}}(\mathcal{V})_S \cong H_1^{\text{FI}}(\mathcal{V})_S \cong 0 \quad \text{for } |S| > r.$$

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Presentation degree controls the degree of generation of the terms in a 2-step resolution of  $\mathcal{V}$  by “free” FI-modules; see Church–Ellenberg [CE17, Proposition 4.2]. Let  $F(M)$  be the contravariant functor from FI to the category of spaces sending a finite set  $S$  to the space  $F_S(M)$  of embeddings of  $S$  into  $M$ . For the set  $[k] := \{1, 2, \dots, k\}$ , the space  $F_{[k]}(M)$  is canonically identified with the ordered configuration space of  $M$ ,

$$F_k(M) = \{(m_1, m_2, \dots, m_k) \in M^k \mid m_i \neq m_j \text{ for all } i \neq j\}.$$

Taking cohomology yields FI-modules  $H^j(F(M))$ . Our main theorem is the following.

**Theorem 1.1.** *Let  $M$  be a connected manifold of dimension  $d = 2$ . Then  $H^j(F(M))$  has generation degree  $\leq 4j + 1$  and presentation degree  $\leq 4j + 2$ . If  $\dim M \geq 3$ , then  $H^j(F(M))$  has generation degree  $\leq 2j + 1$  and presentation degree  $\leq 2j + 2$ .*

Theorem 1.1 implies that, for  $k \geq 4j + 1$ , the group  $H^j(F_k(M))$  is spanned by the  $S_k$ -orbit of the image of  $H^j(F_{4j+1}(M))$  under the map induced by the fibration  $F_k(M) \rightarrow F_{4j+1}(M)$  forgetting all but the first  $(4j + 1)$  points. In other words, every cohomology class in  $H^j(F_k(M))$  can be represented by a sum of classes pulled back along the natural forgetful maps  $F_k(M) \rightarrow F_{4j+1}(M)$ . Moreover, for each  $j$  the sequence of cohomology groups  $\{H^j(F_k(M))\}_k$  is completely determined by the first  $4j + 2$  groups. When  $M$  has dimension at least 3, we obtain the improved result that the sequence of cohomology groups only depends on the first  $2j + 2$  groups. Concretely, Church–Ellenberg [CE17, Theorem C] implies the following corollary.

**Corollary 1.2.** *Let  $M$  be a connected 2-manifold. Then for each homological degree  $j \geq 0$ ,*

$$H^j(F_k(M)) \cong \operatorname{colim}_{\substack{S \subseteq \{1, 2, \dots, k\} \\ |S| \leq 4j+2}} H^j(F_S(M)).$$

*If  $M$  is a connected manifold of dimension  $d \geq 3$ , then for each homological degree  $j \geq 0$ ,*

$$H^j(F_k(M)) \cong \operatorname{colim}_{\substack{S \subseteq \{1, 2, \dots, k\} \\ |S| \leq 2j+2}} H^j(F_S(M)).$$

Another consequence is that the sequences of cohomology groups  $\{H^j(F_k(M))\}_k$  are *centrally stable* in the sense of Putman [Put15]; see also Patz [Pat]. This implies that, for  $k > 4j + 2$ , there is an explicit presentation for the group  $H^j(F_k(M))$  in terms of the groups  $H^j(F_{k-1}(M))$  and  $H^j(F_{k-2}(M))$ , and the natural maps between them [CMNR18, Propositions 2.4]. We therefore obtain recursive descriptions of the cohomology groups  $H^j(F_k(M))$  in the stable range  $k > 4j + 2$ .

See [CE17, Theorem A] and [CMNR18, Propositions 3.1] for more consequences of bounds for generation degree and presentation degree.

Another motivation for seeking improved stable ranges is the notion of *secondary stability* introduced by Galatius–Kupers–Randal-Williams [GKRWa, GKRWb], which was adapted to the representation stability context in Miller–Wilson [MW]. Secondary stability is a pattern in the unstable (co)homology of spaces whose (co)homology exhibits homological or representation stability. A prerequisite for establishing nontrivial secondary stability is sharp stability bounds for primary stability.

**Remark 1.3.** This paper originated as the appendix to Miller–Wilson [MW]. It was later split from that paper for reasons of length. That appendix proved the first explicit bounds for the generation and presentation degree of the integral cohomology of ordered configuration spaces of (possibly closed) manifolds. It was also the first paper to establish stability for configuration spaces of points in non-orientable manifolds. The appendix relied on algebraic results of Church–Ellenberg [CE17]. Since its release, Church–Miller–Nagpal–Reinhold [CMNR18] improved on the stable ranges, using the appendix to address non-orientable manifolds. Bahran [Bah] further improved the stable ranges for high-dimensional orientable manifolds. This updated version of the appendix now relies on algebraic results of Gan–Li [GL], which allow us to establish an even better stable range.

**Remark 1.4.** Church–Miller–Nagpal–Reinhold [CMNR18] described two methods for proving representation stability, which they called Type A and Type B stability setups. Almost all known stability results for FI-modules fall into one of those two categories broadly construed or involve directly calculating the FI-modules in question. Gan–Li [GL] dramatically improved the stable

ranges arising from Type B setups. The methods of this paper can be used to transform most Type A setups currently in the literature into Type B setups, and should result in improved ranges in those situations. In particular, the stable ranges for pure mapping class groups (Jiménez Rolland [JR]), homotopy groups of configuration space (Kupers–Miller [KM18]), configuration spaces of non-manifolds (Tosteson [Tos]), and generalized configuration spaces where collisions are allowed (Petersen [Pet17]) could possibly be improved using the techniques of this paper.

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## 2 Configuration spaces of non-compact manifolds

In this section, we prove a stability theorem for the configuration spaces of a manifold  $M$  in the case that  $M$  is not compact. We begin by recalling the category  $\mathbf{FI}\sharp$  which acts up to homotopy on the configuration spaces of points in a non-compact manifold. The following definition is equivalent to Church–Ellenberg–Farb [CEF15, Definition 4.1.1].

**Definition 2.1.** Let  $\mathbf{FI}\sharp$  denote the category whose objects are based sets and whose morphisms are based maps such that the preimage of every element but the basepoint has size at most one.

We will view  $\mathbf{FI}\sharp$ -modules as  $\mathbf{FI}$ -modules by restricting to the wide subcategory  $\mathbf{FI} \subseteq \mathbf{FI}\sharp$  of injective morphisms. Note that the opposite category of  $\mathbf{FI}\sharp$  is again  $\mathbf{FI}\sharp$ .

When  $M$  is a non-compact manifold, its configuration spaces admit maps  $F_k(M) \rightarrow F_{k+1}(M)$  by introducing a point “at infinity” (see Church–Ellenberg–Farb [CEF15, Section 6.4]), and these maps endow the (co)homology groups with  $\mathbf{FI}\sharp$ -module structures. The following result is due to Church–Ellenberg–Farb [CEF15, Proposition 6.4.2]. Miller–Wilson [MW, Section 3.1] addresses the case that the non-compact manifold  $M$  is not the interior of a compact manifold with boundary.

**Proposition 2.2** (Church–Ellenberg–Farb [CEF15, Proposition 6.4.2]; see also Miller–Wilson [MW, Section 3.1]). *If  $M$  is a non-compact manifold of dimension  $d \geq 2$ , then the assignment  $S \mapsto F_S(M)$  defines a functor from  $\mathbf{FI}\sharp$  to the homotopy category of spaces. In particular, post-composing with the  $j$ th homology or cohomology group functor gives  $\mathbf{FI}\sharp$ -modules which we denote by  $H_j(F(M))$  and  $H^j(F(M))$ , respectively.*

The following result appears in Miller–Wilson [MW, Theorem 3.12 and Theorem 3.27]. It follows easily from Church–Ellenberg–Farb [CEF15, Theorem 6.4.3] if  $M$  is orientable and of finite type.

**Theorem 2.3** ([MW, Theorems 3.12 and 3.27], [CEF15, Theorem 6.4.3]). *Let  $M$  be a non-compact connected manifold of dimension  $d$ . If  $d = 2$ , then  $H_j(F(M))$  has generation degree  $\leq 2j$ . If  $d \geq 3$ , then  $H_j(F(M))$  has generation degree  $\leq j$ .*

Let  $\mathbf{FB}$  denote the category of finite sets and bijections. Let  $\mathbf{Mod}_{\mathbf{FB}}$  denote the category of functors from  $\mathbf{FB}$  to abelian groups, so  $\mathbf{Mod}_{\mathbf{FB}}$  is equivalent to the category of sequences of integral  $S_k$ -representations. Let  $\mathcal{I} : \mathbf{Mod}_{\mathbf{FB}} \rightarrow \mathbf{Mod}_{\mathbf{FI}}$  be the left adjoint of the forgetful functor. Concretely, for an  $\mathbf{FB}$ -module  $\mathcal{U}$ , the  $\mathbf{FI}$ -module  $\mathcal{I}(\mathcal{U})$  is given by the following formula (see Church–Ellenberg–Farb [CEF15, Definition 2.2.2]):

$$\mathcal{I}(\mathcal{U})_S = \bigoplus_{T \subseteq S} \mathcal{U}_T.$$

Using this functor, Church–Ellenberg–Farb [CEF15, Lemma 4.1.5] gave another characterization of  $\mathbf{FI}\sharp$ -modules.

**Theorem 2.4** ([CEF15, Lemma 4.1.5]). *The functor  $\mathcal{I}$  factors through the inclusion  $\mathbf{Mod}_{\mathbf{FI}\sharp} \rightarrow \mathbf{Mod}_{\mathbf{FI}}$  and induces an isomorphism of categories  $\mathbf{Mod}_{\mathbf{FB}} \rightarrow \mathbf{Mod}_{\mathbf{FI}\sharp}$ .*

Church–Ellenberg–Farb [CEF15, Remark 2.3.8] showed that, if we view  $\mathbf{Mod}_{\mathbf{FB}}$  as the codomain of  $H_0^{\mathbf{FI}}$ , then  $H_0^{\mathbf{FI}}$  is a right inverse to  $\mathcal{I}$ .

For an  $\mathbf{FI}$ -module or  $\mathbf{FB}$ -module  $\mathcal{U}$ , we say  $\deg \mathcal{U} \leq d$  if  $\mathcal{U}_k \cong 0$  for  $k > d$ . In particular, the generation degree of  $\mathcal{I}(\mathcal{U})$  is equal to  $\deg \mathcal{U}$ .

Given an  $\mathbf{FI}\sharp$ -module  $\mathcal{V}$ , we can form  $\mathbf{FI}\sharp$ -modules  $\mathrm{Ext}_{\mathbb{Z}}^i(\mathcal{V}, \mathbb{Z})$  by composing with the Ext functors pointwise. Since the opposite category of  $\mathbf{FB}$  is  $\mathbf{FB}$ , we can also perform this construction with  $\mathbf{FB}$ -modules to obtain  $\mathbf{FB}$ -modules.

**Lemma 2.5.** *Let  $\mathcal{V}$  be an  $\mathrm{Fl}_\#^i$ -module with generation degree  $\leq g$ . Then the  $\mathrm{Fl}_\#^i$ -modules  $\mathrm{Ext}_{\mathbb{Z}}^i(\mathcal{V}, \mathbb{Z})$  have generation degree  $\leq g$ .*

*Proof.* Since the direct sum of exact chain complexes of free abelian groups is an exact chain complex of free abelian groups, it follows from the description of  $\mathcal{I}$  above (quoting [CEF15, Definition 2.2.2]) that  $\mathcal{I}$  takes pointwise free resolutions to pointwise free resolutions. Similarly, since  $\mathrm{Hom}$  commutes with finite direct sums, it follows that  $\mathcal{I}$  commutes with pointwise  $\mathrm{Hom}$  functors. Thus  $\mathcal{I}$  commutes with pointwise  $\mathrm{Ext}$  functors and we have isomorphisms

$$\mathrm{Ext}_{\mathbb{Z}}^i(\mathcal{V}, \mathbb{Z}) \cong \mathrm{Ext}_{\mathbb{Z}}^i(\mathcal{I}(H_0^{\mathrm{Fl}}(\mathcal{V})), \mathbb{Z}) \cong \mathcal{I}(\mathrm{Ext}_{\mathbb{Z}}^i(H_0^{\mathrm{Fl}}(\mathcal{V}), \mathbb{Z})).$$

To check that these isomorphisms of abelian groups assemble to form an isomorphism of  $\mathrm{Fl}_\#^i$ -modules, it is helpful to use functorial free resolutions like the bar resolution to compute  $\mathrm{Ext}$ . This isomorphism shows that the generation degree of  $\mathrm{Ext}_{\mathbb{Z}}^i(\mathcal{V}, \mathbb{Z})$  is bounded by  $\deg H_0^{\mathrm{Fl}}(\mathcal{V})$  which is the generation degree of  $\mathcal{V}$ .  $\square$

This gives the following corollary.

**Corollary 2.6.** *Let  $M$  be a non-compact connected manifold of dimension  $d$ . If  $d = 2$ , then  $H^j(F(M))$  has generation degree  $\leq 2j$ . If  $d \geq 3$ , then  $H^j(F(M))$  has generation degree  $\leq j$ .*

*Proof.* By the universal coefficient theorem, there is a short exact sequence of  $\mathrm{Fl}_\#^i$ -modules

$$0 \rightarrow \mathrm{Ext}_{\mathbb{Z}}^1(H_{j-1}(F(M), \mathbb{Z})) \rightarrow H^j(F(M)) \rightarrow \mathrm{Ext}_{\mathbb{Z}}^0(H_j(F(M)), \mathbb{Z}) \rightarrow 0.$$

This gives an exact sequence

$$H_0^{\mathrm{Fl}}(\mathrm{Ext}_{\mathbb{Z}}^1(H_{j-1}(F(M), \mathbb{Z}))) \rightarrow H_0^{\mathrm{Fl}}(H^j(F(M))) \rightarrow H_0^{\mathrm{Fl}}(\mathrm{Ext}_{\mathbb{Z}}^0(H_j(F(M)), \mathbb{Z})).$$

The claim now follows by [Theorem 2.3](#) and [Lemma 2.5](#).  $\square$

**Remark 2.7.** We remark that, when  $M$  is finite type, then [Corollary 2.6](#) is a consequence of the result of Church–Ellenberg–Farb [CEF15, proof of Theorem 4.1.7] that an  $\mathrm{Fl}_\#^i$ -module  $\mathcal{V}$  is finitely generated in degree  $\leq g$  if and only if  $\mathcal{V}_k$  is generated by  $O(k^g)$  elements. We can then use [Theorem 2.3](#) and the universal coefficients theorem to bound the growth in the number of generators of the abelian group  $H^j(F_k(M))$ , and [Corollary 2.6](#) follows.

Higher FI-homology groups of the cohomology groups of configuration spaces of non-compact manifolds vanish uniformly by the following theorem.

**Theorem 2.8** ([CE17, Lemma 2.3]). *Let  $\mathcal{V}$  be an  $\mathrm{Fl}_\#^i$ -module. Then  $H_i^{\mathrm{Fl}}(\mathcal{V}) = 0$  for all  $i > 0$ .*

### 3 The puncture resolution

We now recall the *puncture resolution* from Randal-Williams [RW13], which will allow us to leverage results for non-compact manifolds to prove results for compact manifolds.

**Definition 3.1.** Fix a finite set  $S$ . We define the *puncture resolution* of the configuration space  $F_S(M)$  as follows. For  $p \geq -1$ , denote the topological spaces

$$\mathrm{Pun}_p(F_S(M)) = \bigsqcup_{(x_0, \dots, x_p) \in F_{\{0, \dots, p\}}(M)} F_S(M - \{x_0, \dots, x_p\}).$$

In particular,  $\mathrm{Pun}_{-1}(F_S(M)) = F_S(M)$ . See [Figure ??](#) for an illustration. For fixed  $S$ , the spaces  $\mathrm{Pun}_p(F_S(M))$  assemble to form an augmented semi-simplicial space. The  $i$ th face map is induced by the inclusion

$$M - \{x_0, \dots, x_p\} \hookrightarrow M - \{x_0, \dots, \hat{x}_i, \dots, x_p\}.$$

Given an injection  $S \hookrightarrow T$ , composition of embeddings gives a map of augmented semi-simplicial spaces  $\mathrm{Pun}_\bullet(F_T(M)) \rightarrow \mathrm{Pun}_\bullet(F_S(M))$ . In this way, the puncture resolution assembles to form an augmented semi-simplicial co-FI-space  $\mathrm{Pun}_\bullet(F(M))$ . Thus the cohomology groups of the space of  $p$ -simplices of the puncture resolution assemble to form an FI-module and the face maps are maps of FI-modules. In particular, the action of the symmetric group  $S_k$  on  $F_k(M)$  extends to an action on  $\mathrm{Pun}_\bullet(F_k(M))$ . Randal-Williams proved the following [RW13, Section 9.4].

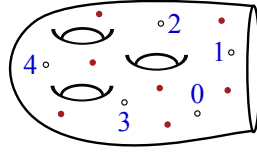


Figure 1: An element of  $\text{Pun}_4(F_7(M))/S_7$ .

**Theorem 3.2** ([RW13, Section 9.4]). *Let  $M$  be a manifold. The augmentation map*

$$\|\text{Pun}_\bullet(F_k(M))\|/S_k \rightarrow F_k(M)/S_k$$

*is a weak equivalence.*

We now use [Theorem 3.2](#) to prove the following.

**Proposition 3.3.** *Let  $M$  be a manifold. The augmentation map  $\|\text{Pun}_\bullet(F_k(M))\| \rightarrow F_k(M)$  is a weak equivalence.*

*Proof.* We proceed as in the proof of Miller–Wilson [[MW](#), Proposition 3.8]. It suffices to show that the homotopy fiber of the map  $\|\text{Pun}_\bullet(F_k(M))\| \rightarrow F_k(M)$  is contractible. If we use the standard path space construction of homotopy fibers, then this homotopy fiber is homeomorphic to the homotopy fiber of  $\|\text{Pun}_\bullet(F_k(M))\|/S_k \rightarrow F_k(M)/S_k$ , which is contractible by [Theorem 3.2](#).  $\square$

## 4 Proof of [Theorem 1.1](#)

In this section, we prove the main theorem. We begin by recalling the definition of FI-hyperhomology.

**Definition 4.1.** Given an FI-chain complex  $\mathcal{V}_*$ , we denote the  $i$ th hyperhomology group associated to the chain complex  $\mathcal{V}_*$  and the functor  $H_0^{\text{FI}}$  by  $\mathbf{H}_i^{\text{FI}}(\mathcal{V}_*)$  and refer to it as FI-hyperhomology.

A key component of our proof is the following result of Gan–Li.

**Theorem 4.2** (Gan–Li [[GL](#), Theorem 5]). *Let  $\mathcal{V}_*$  be an FI-chain complex supported in nonnegative homological degrees. Its  $i$ th homology group is an FI-module  $H_i(\mathcal{V}_*)$ . Let*

$$\mathbf{t}_j(\mathcal{V}_*) = \deg \mathbf{H}_j^{\text{FI}}(\mathcal{V}_*)$$

*denote the degree of the FI-hyperhomology of  $\mathcal{V}_*$  in homological degree  $j$ . Then for each  $i \geq 0$ ,*

$$\begin{aligned} \deg H_0^{\text{FI}}(H_i(\mathcal{V}_*)) &\leq 2\mathbf{t}_i(\mathcal{V}_*) + 1, \\ \deg H_1^{\text{FI}}(H_i(\mathcal{V}_*)) &\leq 2 \max(\mathbf{t}_i(\mathcal{V}_*), \mathbf{t}_{i+1}(\mathcal{V}_*)) + 2. \end{aligned}$$

The following result is well-known; see for example Bendersky–Gitler [[BG91](#), Proposition 1.2].

**Proposition 4.3** (E.g., Bendersky–Gitler [[BG91](#), Proposition 1.2]). *Let  $X_\bullet$  be a semi-simplicial space. Let  $X^{*,*}$  be the double complex with  $X^{p,q} = C^q(X_p)$  with one differential given by the differential on singular cochains and the other given by the alternating sum of the facemaps. Let  $X^*$  be the total complex of  $X^{*,*}$ . Then  $H_i(X^*) \cong H^i(\|X_\bullet\|)$ .*

We now prove the main theorem, an improved stable range for the cohomology of ordered configuration spaces.

*Proof of [Theorem 1.1](#).* Gan–Li’s result [Theorem 4.2](#) requires an FI-chain complex supported in nonnegative degrees, so we cannot apply the result directly to the cochain complex  $C^*(\text{Pun}_\bullet(F(M)))$ . To produce a complex satisfying Gan–Li’s assumptions, we re-index and truncate this cochain complex. For fixed  $N \in \mathbb{N}_{\geq 1}$ , define the homologically-graded FI-chain complex

$$C_i^{\leq N} := \begin{cases} \bigoplus_{s+t=N-i} C^t(\text{Pun}_s(F(M))), & 0 \leq i \leq N \\ 0, & \text{otherwise.} \end{cases}$$

By [Proposition 3.3](#) and [Proposition 4.3](#), we see that  $H_i(C_*^{\leq N})_k \cong H^{N-i}(F_k(M))$  for  $1 \leq i \leq N$ .

The complex  $C_*^{\leq N}$  is the total complex of a double complex with vertical chain complexes  $C^{N-s-*}(\text{Pun}_s(F(M)))$  for each  $s \geq 0$ . So now fix  $s$  and consider the FI-hyperhomology associated to the chain complex  $C^{N-s-*}(\text{Pun}_s(F(M)))$ . One of the two standard spectral sequences converging to the hyperhomology groups  $\mathbf{H}_{p+q}^{\text{FI}}(C^{N-s-*}(\text{Pun}_s(F(M))))$  has  $E^2$  page

$$\begin{aligned} E_{p,q}^2 &= H_p^{\text{FI}}(H_q(C^{N-s-*}(\text{Pun}_s(F(M)))))) \\ &\cong \bigoplus_{\substack{(x_0, \dots, x_s) \text{ in} \\ F_{\{0, \dots, s\}}(M)}} H_p^{\text{FI}}(H^{N-s-q}(F(M - \{x_0, \dots, x_s\}))) \end{aligned}$$

Observe that punctured manifolds are not compact, so by [Proposition 2.2](#) the cohomology groups of their configuration spaces form  $\text{FI}_\#$ -modules. Then by Church–Ellenberg’s result [Theorem 2.8](#),

$$E_{p,q}^2 \cong \begin{cases} \bigoplus_{\substack{(x_0, \dots, x_s) \text{ in} \\ F_{\{0, \dots, s\}}(M)}} H_0^{\text{FI}}(H^{N-s-q}(F(M - \{x_0, \dots, x_s\}))), & p = 0 \\ 0, & p > 0. \end{cases}$$

Thus the spectral sequence collapses at  $E^2$  and we find

$$\mathbf{H}_q^{\text{FI}}(C^{N-s-*}(\text{Pun}_s(F(M)))) \cong \bigoplus_{\substack{(x_0, \dots, x_s) \text{ in} \\ F_{\{0, \dots, s\}}(M)}} H_0^{\text{FI}}(H^{N-s-q}(F(M - \{x_0, \dots, x_s\}))).$$

By [Corollary 2.6](#),

$$\begin{aligned} \mathbf{t}_q^N(s) &:= \deg \mathbf{H}_q^{\text{FI}}(C^{N-s-*}(\text{Pun}_s(F(M)))) \\ &= \deg H_0^{\text{FI}}(H^{N-s-q}(F(M - \{x_0, \dots, x_s\}))) \quad \text{for } (x_0, \dots, x_s) \in F_{\{0, \dots, s\}}(M), \\ &\leq \begin{cases} 2N - 2s - 2q, & \dim(M) = 2 \\ N - s - q, & \dim(M) \geq 3. \end{cases} \end{aligned}$$

The complex  $C_*^{\leq N}$  is an iterated extension of the complexes  $C^{N-s-*}(\text{Pun}_s(F(M)))$  for  $0 \leq s \leq N$ . Concretely, we can by definition express  $C_*^{\leq N}$  as

$$C_*^{\leq N} = \bigoplus_{\substack{0 \leq i \leq N \\ 0 \leq s \leq N-i}} C^{N-i-s}(\text{Pun}_s(F(M))).$$

We can then define a filtration of  $C_*^{\leq N}$  by an index  $S$ , bounding the simplicial variable  $s$ ,

$$\begin{aligned} C_*^{\leq N}(S) &= \bigoplus_{\substack{0 \leq i \leq N \\ 0 \leq s \leq \min(S, N-i)}} C^{N-i-s}(\text{Pun}_s(F(M))) \\ &= \bigoplus_{\substack{0 \leq s \leq S \\ 0 \leq i \leq N-s}} C^{N-i-s}(\text{Pun}_s(F(M))) \end{aligned}$$

and so for each fixed  $0 \leq S \leq N$  we find

$$C_*^{\leq N}(S)/C_*^{\leq N}(S-1) \cong \bigoplus_{0 \leq i \leq N-S} C^{N-i-S}(\text{Pun}_S(F(M))).$$

By (for example) Weibel [[Wei95](#), Lemma 5.7.5], given a short exact sequence  $0 \rightarrow K_* \rightarrow A_* \rightarrow Q_* \rightarrow 0$  of bounded-below FI-chain complexes, there is a long exact sequence of FI-hyperhomology groups

$$\cdots \rightarrow \mathbf{H}_{q+1}^{\text{FI}}(Q_*) \rightarrow \mathbf{H}_q^{\text{FI}}(K_*) \rightarrow \mathbf{H}_q^{\text{FI}}(A_*) \rightarrow \mathbf{H}_q^{\text{FI}}(Q_*) \rightarrow \cdots$$

Thus by induction on  $s$ , the degree  $\mathbf{t}_q^N := \deg \mathbf{H}_q^{\text{FI}}(C_*^{\leq N})$  is bounded above by  $\max_{0 \leq s \leq N} \mathbf{t}_q^N(s)$ . Specifically,

$$\mathbf{t}_q^N = \deg \mathbf{H}_q^{\text{FI}}(C_*^{\leq N}) \leq \begin{cases} 2N - 2q, & \dim(M) = 2 \\ N - q, & \dim(M) \geq 3. \end{cases}$$

Our goal is to find bounds for the generation and presentation degrees of the FI-module  $H^j(F(M))$ . As noted earlier,  $H_i(C_*^{\leq N})_k \cong H^{N-i}(F_k(M))$  for  $1 \leq i \leq N$ . In particular, we have

$$H^j(F(M)) = H_1(C_*^{\leq(j+1)}).$$

Using Gan–Li’s result [Theorem 4.2](#), we obtain the following:

$$\begin{aligned} \deg H_0^{\text{FI}}(H^j(F(M))) &= \deg H_0^{\text{FI}}(H_1(C_*^{\leq(j+1)})) \\ &\leq 2\mathbf{t}_1^{j+1} + 1 \\ &\leq \begin{cases} 4j + 1, & \dim(M) = 2 \\ 2j + 1, & \dim(M) \geq 3 \end{cases} \end{aligned}$$

$$\begin{aligned} \deg H_1^{\text{FI}}(H^j(F(M))) &= \deg H_0^{\text{FI}}(H_1(C_*^{\leq(j+1)})) \\ &\leq 2 \max(\mathbf{t}_1^{j+1}, \mathbf{t}_2^{j+1}) + 2 \\ &\leq \begin{cases} 4j + 2, & \dim(M) = 2 \\ 2j + 2, & \dim(M) \geq 3. \end{cases} \end{aligned}$$

This concludes the proof. □

In the above proof, we considered  $H_1(C_*^{\leq(j+1)}) \cong H^j(F_k(M))$  for concreteness. The proof would have worked equally well by instead studying  $H_i(C_*^{\leq(j+i)})_k \cong H^j(F_k(M))$  for any fixed  $i \geq 1$ .

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