Topic Proposal
The Cohomology of Configuration Space and Representation Stability

Jenny Wilson
Discussed with Prof. Benson Farb

Winter 2010-2011

The theory of the configuration space of a manifold provides illustrative examples of a number of topological phenomena. We begin with a discussion of the configuration space of the complex plane, and give a classical description of the braid and pure braid groups. We next give an alternative interpretation of the braid groups as mapping classes of punctured disks, an interpretation which provides a tool in the study of the mapping class groups of higher genus surfaces. Using the Leray-Serre spectral sequence described by Arnol'd, we then give a presentation for the rational cohomology of the pure braid group, which we use to motivate a discussion of the new theory of representation stability developed by Church and Farb – a notion of stability for a sequence of groups with S_n-actions, in terms of their decomposition into irreducible $\mathbb{Q}S_n$-modules. We prove this property for the rational cohomology of the pure braid group. Finally, we outline Totaro's work on the configuration space of a real oriented m-manifold X, which demonstrates that the cohomology of a manifold determines the cohomology of its configuration space. Totaro exhibits a Leray spectral sequence converging to these cohomology groups, and, in the case that X is a complex projective variety, he gives an explicit computation of the cohomology ring of the configuration space.

1 The Braid Group and the Pure Braid Group

1.1 Some Characterizations of the Braid Group

Definition 1.1. For a topological space X, define $\mathcal{F}(X,n) = \{(z_1, z_2, \dots, z_n) \in X^n \mid z_i \neq z_j \text{ for } i \neq j\}$, the space of ordered n-tuples of distinct points in X. The symmetric group S_n acts on by $\mathcal{F}(X,n)$ by permuting the points; the orbit space $\mathcal{C}(X,n) = \mathcal{F}(X,n)/S_n$ is called the ***configuration space*** of n (unordered) points in X. Its cover $\mathcal{F}(X,n)$ is sometimes called the ***ordered*** or ***coloured*** configuration space of n points.

Definition 1.2. The ***braid group*** B_n is the fundamental group of $\mathcal{C}(\mathbb{C},n)$, the configuration space of n complex points. The braid group is also known by its pictorial description as braids in n disjoint strands, rooted in the plane, under concatenation. The ***pure braid group*** P_n, also called the ***coloured braid group***, is the subgroup of B_n preserving the order of the n strands. The pure braid group may be defined as the fundamental group of $\mathcal{F}(\mathbb{C},n)$; equivalently, it is the kernel of the natural map $B_n \twoheadrightarrow S_n$.

An alternate definition for the braid group is the mapping class group of the n-puncture closed 2-disk D_n;

$$B_n \cong \mathrm{Mod}(D_n) := \pi_0(\mathrm{Homeo}^+(D_n, \partial D_n))$$

and the pure braid group $P_n \cong \mathrm{PMod}(D_n)$ is the group of mapping classes that fix the punctures pointwise.

This description of B_n can be derived using a generalized form of the Birman exact sequence,

$$1 \longrightarrow \pi_1\big(\mathcal{C}(D,n)\big) \xrightarrow{Push} \mathrm{Mod}(D_n) \xrightarrow{Forget} \mathrm{Mod}(D) \longrightarrow 1.$$

Here D denotes the closed 2-disk, which (by the Alexander Lemma) has trivial mapping class group.

1.2 The Structure of the Braid Groups

We now give presentations for the braid pure braid group, and highlight some algebraic properties.

The braid group B_n is generated by the braids σ_i, given by passing strand $i+1$ over top of strand i. The group is finitely presented:

$$B_n = \big\langle \sigma_1, \ldots, \sigma_{n-1} \;\big|\; \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \text{ for } i < (n-1),\ \sigma_i\sigma_j = \sigma_j\sigma_i \text{ for } |i-j| > 1 \big\rangle$$

where the relationship $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ is called the ***braid relation***.

From this presentation, we see that abelianization of B_n is isomorphic to $\mathbb{Z}$; the braid relation implies that all generators σ_i must be identified as a single cyclic generator. The centre $Z(B_n) \cong \mathbb{Z}$ is cyclically generated by $z = (\sigma_1 \cdots \sigma_{n-1})^n$. As a braid, this is a counterclockwise 2π twist of the n strands – as a mapping class, this is the Dehn twist around the disk's boundary. The quotient $B_n/Z(B_n)$ is the mapping class group of the n-puncture plane, as we may see by "capping" the boundary of D_n by a 1-puncture disk.

It is a difficult result that the braid group is torsion-free. Considering B_n as the mapping class group of a surface with boundary, this result may be viewed as a consequence of the Nielsen realization theorem. Any torsion subgroup of $Mod(D_n)$ would be the isomorphic image of an isometry group of some hyperbolic metric on D_n fixing the boundary, and so must be trivial.

From the short exact sequence $1 \to P_n \to B_n \to S_n \to 1$ and the presentation for the symmetric group S_n

$$S_n = \big\langle s_1, \ldots, s_{n-1} \;\big|\; s_is_j = s_js_i \text{ for } |i-j| > 1,\ s_is_{i+1}s_i = s_{i+1}s_is_{i+1},\ s_i^2 = 1 \text{ for all } i \big\rangle$$

we deduce that the pure braid group is generated by preimages of conjugates of the relation s_i^2, that is, elements $T_{i,j} = T_{j,i}$ wrapping the i^{th} strand counterclockwise around the j^{th} strand.

Explicitly, P_n has presentation:

$$\begin{aligned} P_n = \big\langle T_{i,j} \text{ for } i \neq j, 1 \leq i,j \leq n \;\big|\; & T_{p,q}T_{r,s} = T_{r,s}T_{p,q}, \quad T_{p,s}T_{q,r} = T_{q,r}T_{p,s}, \\ & T_{p,r}T_{q,r}T_{p,q} = T_{q,r}T_{p,q}T_{p,r} = T_{p,q}T_{p,r}T_{q,r}, \\ & T_{q,s}\big(T_{r,s}T_{q,r}T_{r,s}^{-1}\big) = \big(T_{r,s}T_{q,r}T_{r,s}^{-1}\big)T_{q,s} \quad \text{for } p<q<r<s \big\rangle \end{aligned}$$

All relations belong to the commutator subgroup of P_n, so the abelianization of P_n is the free abelian group generated by the $T_{i,j}$ – a signed count of the number of times the i^{th} strand wraps around the j^{th}. The generator z of the centre of B_n is an element of P_n, and $Z(P_n) = Z(B_n)$. The quotient $P_n/Z(P_n)$ can be identified with the PMod($S_{0,n+1}$), the mapping classes of the n-puncture plane fixing the punctures pointwise, and P_n splits over the quotient $P_n \cong P_n/Z(P_n) \times \mathbb{Z}$.

An alternate construction for P_n comes from the short exact sequence $1 \to F_n \to P_{n+1} \to P_n \to 1$ given by "forgetting" strand $n+1$. The kernel F_n is the free group generated by the elements $T_{i,n+1}$. Since the addition of a strand to P_n gives a splitting map $P_n \to P_{n+1}$, we can write P_n inductively as an extension of free groups.

2 Braid Groups in the Study of Mapping Classes

The identification of the braid group with Mod(D_n) gives us a tool to study the mapping class groups of higher genus surfaces: by defining a double cover of D_n by surfaces with 1 or 2 boundary components, we can deduce relations between the generators of the mapping class groups of those surfaces.

Definition 2.1. Denote by $\mathcal{S}_g^b$ the genus g surface with b boundary components, and let $\mathcal{S}_g$ denote $\mathcal{S}_g^0$. Given an order-2 element $i \in \text{Homeo}^+(\mathcal{S}_g^1)$ with $2g+1$ fixed points (analogous to the hyperelliptic involution of $\mathcal{S}_g$) we define the ***symmetric homeomorphisms SHomeo***$^+(\mathcal{S}_g^1)$ to be the i-invariant subgroup of $\text{Homeo}^+(\mathcal{S}_g^1)$. This subgroup descends to the quotient of $\mathcal{S}_g^1$ by i, a disk with boundary, and stabilizes the set of $(2g+1)$ cone points.

Definition 2.2. The ***symmetric mapping class group SMod***$(\mathcal{S}_g^1)$ is the subgroup of $\text{Mod}(\mathcal{S}_g^1)$ corresponding to the symmetric homeomorphisms, that is, $\text{SHomeo}^+(\mathcal{S}_g^1)$ up to isotopy. It is a difficult result of Birman and Hilden that any two isotopic symmetric homeomorphisms are in fact isotopic through symmetric homeomorphisms, so

$$\text{SMod}(\mathcal{S}_g^1) \cong \pi_0\big(\text{SHomeo}^+(\mathcal{S}_g^1)\big),$$

and, consequently, that $\text{SMod}(\mathcal{S}_g^1) \cong B_{2g+1}$.

By choosing an involution i of the genus g surface $\mathcal{S}_g^2$ with 2 boundary components, such that i has $2g+2$ fixed points that interchanges the two boundary components, we can similarly define $\text{SMod}(\mathcal{S}_g^2)$, and again the work of Birman and Hilden shows $\text{SMod}(\mathcal{S}_g^2) \cong B_{2g+2}$.

There are known presentations for the mapping class groups of $\mathcal{S}_g$ and $\mathcal{S}_g^1$, finitely generated by Dehn twists. A number of relations can be deduced from relations in the braid group, including the braid relation and the so-called k-chain relations. The better-understand structure of the braid groups then gives a means of deriving, and geometric interpretations for, these algebraic properties of the mapping class groups.

3 The Cohomology of the Pure Braid Group

We will now sketch a proof of an explicit presentation for the cohomology ring $H^*(P_n;\mathbb{Q})$, due to Arnol'd.

Theorem 3.1. *The rational cohomology ring $H^*(P_n;\mathbb{Q})$ is a quotient of the exterior algebra*

$$H^*(P_n;\mathbb{Q}) = \Lambda^*[\omega_{i,j}]/\langle R_{i,j,k}\rangle \quad \textit{for distinct } i,j,k \textit{ with } 1 \leq i,j,k \leq n$$

where $R_{i,j,k} = \omega_{i,j}\omega_{j,k} + \omega_{j,k}\omega_{k,i} + \omega_{k,i}\omega_{i,j}$, and the cocycles $\omega_{i,j}$ are dual to the generators $T_{i,j}$ of P_n.

The map $\mathcal{F}(\mathbb{C},n) \to \mathcal{F}(\mathbb{C},n-1)$ obtained by "forgetting" the last point gives a fibre bundle with fibres $F \cong \mathbb{C}\backslash\{z_1,\ldots,z_{n-1}\}$ (for some $(n-1)$ points $z_1,\ldots,z_{n-1} \in \mathbb{C}$). An induction argument on the associated long exact sequence on homotopy groups shows that $\mathcal{F}(\mathbb{C},n)$ is a $K(P_n,1)$ spaces, so we can identify the cohomology of P_n with $H^*(\mathcal{F}(\mathbb{C},n))$.

Arnol'd proved the decomposition of groups

$$H^*(P_n) = H^*(S^1) \otimes H^*(S^1 \vee S^1) \otimes \cdots \otimes H^*\Big(\bigvee^{n-1} S^1\Big)$$

using an induction argument on the Leray-Serre spectral sequence for the fibration $\mathcal{F}(\mathbb{C},n) \to \mathcal{F}(\mathbb{C},n-1)$. Because the fibration has a section, the sequence degenerates at the E_2 page, giving

$$H^k(P_n) = \bigoplus_{p+q=k} H^p\Big(P_{n-1};\ H^q\Big(\bigvee^{n-1} S^1\Big)\Big) = \bigoplus_{p+q=k} H^p(P_{n-1}) \otimes H^q\Big(\bigvee^{n-1} S^1\Big)$$

where we may identify the generators of $H^q(\bigvee^{n-1} S^1)$ with the classes $\omega_{1,n},\ldots,\omega_{n-1,n}$. It follows inductively that

$$H^k(P_n) \cong \{\omega_{i_1,j_1} \wedge \omega_{i_2,j_2} \wedge \cdots \wedge \omega_{i_k,j_k} \mid j_1 < j_2 < \cdots < j_k, \text{ and } i_m < j_m \text{ for all } 1 \leq m \leq k\} \text{ as a group.}$$

It remains to verify the ring structure. The relation $R_{i,j,k}$ can be checked directly, for example, by viewing $\mathcal{F}(\mathbb{C},n)$ as a submanifold of $\mathbb{C}^n$, and identifying the elements $\omega_{i,j}$ as the de Rham cocycles $\omega_{i,j} = \frac{dz_i - dz_j}{z_i - z_j}$. This gives a map $\big(\Lambda^*[\omega_{i,j}]/\langle R_{i,j,k}\rangle\big) \longrightarrow H^*(P_n)$, which surjects by a dimension count, and so concludes the proof.

4 The Cohomology of P_n is Uniformly Representation Stable

4.1 Representation Stability

Though the cohomology of the pure braid group $H^*(P_n;\mathbb{Q})$ is not stable in the classical sense, we can consider its structure as a $\mathbb{Q}S_n$-module, and give a notion of 'stability' in terms of its decomposition into irreducible S_n-representations. To formalize this idea, we present the theory of 'representation stability', developed by Church and Farb, as it applies to the symmetric group.

The representation theory of the symmetric group S_n over a characteristic-zero field K is well understood. According to Maschke's theorem, the group algebra KS_n is semi-simple; any finite dimensional representation V may be written as a direct sum of irreducible representations. The irreducible representations of S_n are classified by partitions of n, depicted as Young tableaux. In brief, for $K = \mathbb{Q}$, the irreducible representation corresponding to a given Young tableau λ can be identified as a subrepresentation of the regular representation $\mathbb{Q}S_n$ – it is the $\mathbb{Q}S_n$-span of a particular element $a_\lambda \in \mathbb{Q}S_n$ associated to λ, called the Young symmetrizer.

The classification of the irreducible representations by partitions of n gives a natural identification of irreducible S_n-representations with irreducible representations of S_m for any $m \geq n$. Specifically, for a partition λ of n, say, $a_0 + \cdots + a_r = n$ where $a_0 \geq \cdots \geq a_r$, we denote the associated irreducible S_n-representation $V(\lambda) = V(a_1, \ldots, a_r)$ By omitting a_0, this notation suppresses the dependence on n. Then, for any $m \geq n$, $V(a_1, \ldots, a_r)$ is associated to the irreducible S_m-representation corresponding to the partition $\left(m - \sum_{i=1}^r a_i\right) + a_1 + \cdots + a_r$ of m. These identifications allow us to define representation stability:

Definition 4.1. A sequence $\{V_n\}_{n\in\mathbb{N}}$ of S_n-representations with maps $\phi_n : V_n \to V_{n+1}$ is called ***consistent*** if ϕ_n is compatible with the S_n-action in the sense that the following diagram commutes:

$$\begin{array}{ccc} V_n & \xrightarrow{\phi_n} & V_{n+1} \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g'} \\ V_n & \xrightarrow{\phi_n} & V_{n+1} \end{array}$$

where g' is the image of g under the natural inclusion $S_n \hookrightarrow S_{n+1}$.

Definition 4.2. A consistent sequence $\{V_n\}_{n\in\mathbb{N}}$ of S_n-representations as above is called ***representation stable*** if it satisfies the following properties:
I. ***Injectivity.*** The maps $\phi_n : V_n \to V_{n+1}$ inject, for n sufficiently large,.
II. ***Surjectivity.*** The space V_{n+1} is spanned by S_{n+1} orbit of the image $\phi_n(V_n)$, for n sufficiently large.
III. ***Multiplicities.*** In the decomposition $V_n = \bigoplus_\lambda c_{\lambda,n} V(\lambda)$ of V_n into irreducible S_n-representations, for each λ, the coefficient $c_{\lambda,n}$ is eventually independent of n.

The representation stable sequence of S_n-representations is ***uniformly*** representation stable if the multiplicities $c_{\lambda,n}$ stabilize at some $N \in \mathbb{N}$ not depending on λ.

We now have the groundwork laid to prove the representation stability of the $H^*(P_n;\mathbb{Q})$.

4.2 A Proof of Representation Stability for the Pure Braid Group

The rational cohomology of the pure braid group $H^*(P_n;\mathbb{Q})$ admits an action of the symmetric group S_n by permuting indices, induced by the action of B_n on P_n by conjugation; $\sigma \cdot \omega_{i,j} = \omega_{\sigma(i),\sigma(j)}$ for $\sigma \in S_n$. We will outline a proof, due to Church and Farb, that the sequence $\{H^*(P_n;\mathbb{Q})\}_{n\in\mathbb{N}}$ is uniformly representation stable with respect to this action, and the inclusions $\phi_n : H^*(P_n;\mathbb{Q}) \to H^*(P_{n+1};\mathbb{Q})$ induced by the projection $\mathcal{F}(\mathbb{C}, n+1) \to \mathcal{F}(\mathbb{C}, n)$ onto the first n coordinates.

Theorem 4.3. *For $k > 0$, the sequence $\{H^k(P_n;\mathbb{Q})\}$ of S_n-representations is representation stable, uniformly for $n \geq 4k$.*

The maps ϕ_n take $\omega_{i,j}$ in $H^1(P_n;\mathbb{Q})$ to $\omega_{i,j}$ in $H^1(P_{n+1};\mathbb{Q})$. Since the projection $\mathcal{F}(\mathbb{C}, n+1) \to \mathcal{F}(\mathbb{C}, n)$ has a section, we see that the maps inject. We further see that they satisfy the surjectivity criterion: for $n \geq 2k$, any basis element of $H^i(P_{n+1};\mathbb{Q})$ cannot have all indices $\{1, \ldots, n, n+1\}$, so acting by a suitable choice of permutation in S_{n+1} gives an element of $\mathrm{Im}(\phi_n)$.

To prove the multiplicities criterion, we invoke a combinatorial result by Hemmer. Recall that, given a subgroup H of a finite group G and an H-representation V, the ***induced representation*** $\mathrm{Ind}_H^G(V)$ is the $\mathbb{Q}G$-module $\mathbb{Q}G \otimes_{\mathbb{Q}H} V$, equivalently, the G-representation constructed as a direct sum $\mathrm{Ind}_H^G(V) = \bigoplus_{\sigma \in G/H} \sigma V$ of a translate of V for each coset σH. Recall additionally that, given two groups H_1 and H_2 with respective representations V_1 and V_2, the product $H_1 \times H_2$ acts on the ***external tensor product*** $V_1 \boxtimes V_2$ of V_1 and V_2. In this notation, Hemmer's result states:

Theorem 4.4. *Given $r \geq 1$, any subgroup H of S_r and H-representation V, and given the trivial representation $\mathbb{Q}$ of S_{n-r}, the sequence of S_n-representations $\{\mathrm{Ind}_{H \times S_{n-r}}^{S_n}(V \boxtimes \mathbb{Q})\}_{n \in \mathbb{N}}$ is representation stable, stabilizing uniformly for $n \geq 2r$.*

Our objective is now to show that the cohomology groups of P_n have this structure.

Let $[n] = \{1, \ldots, n\}$. Let $\mathscr{P}$ denote a partition of $[n]$, and let $\overline{\mathscr{P}}$ denote the associated partition of n. Each block of size j defines a projection $P_n \to P_j$; the partition $\mathscr{P}$ defines a map from P_n to the 'Young subgroup' $P_{\mathscr{P}}$ of P_n, a product of pure braid groups over the blocks of $\mathscr{P}$. This map induces an inclusion $H^*(P_{\mathscr{P}};\mathbb{Q}) \hookrightarrow H^*(P_n;\mathbb{Q})$. If $\mathscr{P}$ has $(n-k)$ blocks, $H^*(P_{\mathscr{P}};\mathbb{Q})$ will have top degree k, and we define $H^{\mathscr{P}}(P_n) \subseteq H^k(P_n;\mathbb{Q})$ to be the image of this top group. The key observation is that $H^*(P_n;\mathbb{Q})$ decomposes into a sum of $\mathbb{Q}S_n$-modules $\bigoplus_{\mathscr{P}} H^{\mathscr{P}}(P_n)$ over all partitions $\mathscr{P}$ of $[n]$.

The symmetric group S_n acts on the set of partitions associated to a given partition λ of n, and accordingly, S_n acts on $\bigoplus_{\{\mathscr{P} | \overline{\mathscr{P}} = \lambda\}} H^{\mathscr{P}}(P_n)$. By an orbit-stabilizer argument, we can identify these S_n-translates of $H^{\mathscr{P}}(P_n)$ with the cosets $S_n/\mathrm{Stab}(\mathscr{P})$, so

$$\bigoplus_{\{\mathscr{P} | \overline{\mathscr{P}} = \lambda\}} H^{\mathscr{P}}(P_n) \cong \mathrm{Ind}_{\mathrm{Stab}(\mathscr{P}')}^{S_n} H^{\mathscr{P}'}(P_n)$$

for some fixed $\mathscr{P}'$ with $\overline{\mathscr{P}'} = \lambda$. Note that, if $\mathscr{P}'$ has m_j blocks of size j, then its stabilizer is $\bigoplus_{j=1}^n \left((S_j)^{m_j} \rtimes S_{m_j}\right)$, where $(S_j)^{m_j}$ permutes the numbers within the blocks of size j, and S_{m_j} permutes the m_j blocks.

For any partition $\mathscr{P}$ of $[n]$, define $\mathscr{P}\langle n+1 \rangle = \mathscr{P} \cup \{n+1\}$. Observe that ϕ_n takes $H^{\mathscr{P}}(P_n)$ isomorphically to $H^{\mathscr{P}\langle n+1 \rangle}(P_n)$. Observe further that, for $n \geq 2k$, any partition of $[n+1]$ must have at least one block of size 1, and so must be in the S_{n+1} orbit of $H^{\mathscr{P}\langle n+1 \rangle}(P_n)$ for some partition $\mathscr{P}$ of $[n]$. We find:

$$H^k(P_{n+1};\mathbb{Q}) \cong \bigoplus_{\substack{\lambda \vdash n \\ \text{with } n-k \text{ blocks}}} \mathrm{Ind}_{\mathrm{Stab}(\mathscr{P}\langle n+1 \rangle)}^{S_{n+1}} H^{\mathscr{P}\langle n+1 \rangle}(P_{n+1}).$$

Finally, observe that, if we write $\mathrm{Stab}(\mathscr{P}) \cong H_{\mathscr{P}} \oplus S_{m_1}$, then $\mathrm{Stab}(\mathscr{P}\langle n+1 \rangle) \cong H_{\mathscr{P}} \oplus S_{m_1+1}$; where the action of the stabilizer on size-one blocks of any partition corresponds to the trivial action on the associated cohomology subspace. We have therefore decomposed $H^*(P_n;\mathbb{Q})$ into subrepresentations of the form necessary to apply Hemmer's result, which proves the theorem.

5 The Configuration Space of an Oriented Manifold

The classical interest in the configuration space of the complex plane motivates the study of the configuration space $\mathcal{F}(X, n)$ of any topological space X. We consider an oriented manifold X of real dimension m. The Leray spectral sequence associated to the inclusion $\mathcal{F}(X, n) \hookrightarrow X^n$ converges to $H^*(\mathcal{F}(X, n);\mathbb{Q})$, and the E_2 page has a simple description. The following theorems are due to Totaro:

Theorem 5.1. *Let $a \neq b \in [n]$, and let $p_a^* : H^*(X) \to H^*(X^n)$ and $p_{a,b}^* : H^*(X^2) \to H^*(X^n)$ be the maps induced by the projections p_a and $p_{a,b}$. Let $\Delta \in H^m(X^2)$ be the class of the diagonal. Then the map $\mathcal{F}(X,n) \hookrightarrow X^n$ determines a Leray spectral sequence converging to $H^*(\mathcal{F}(X,n);\mathbb{Q})$ as an algebra. The E_2 page is a quotient of the algebra $H^*(X^n;\mathbb{Q})[G_{a,b}]$, $a \neq b \in [n]$, by the relations*

$$(0)\ G_{a,b} = (-1)^m G_{a,b}, \qquad (2)\ G_{a,b}G_{a,c} + G_{b,c}G_{b,a} + G_{c,a}G_{c,b} = 0 \quad (a,b,c \text{ distinct}),$$
$$(1)\ (G_{a,b})^2 = 0, \qquad (3)\ p_a^*(x)G_{a,b} = p_b^*(x)G_{a,b} \quad (a \neq b,\ x \in H^*(X;\mathbb{Q})).$$

Here, $H^k(X^n;\mathbb{Q})$ has degree $(k,0)$, and $G_{a,b}$ have degree $(0,m-1)$.

The differential d is given by $d(G_{a,b}) = p_{a,b}^*\Delta$.

Over $\mathbb{Z}$, the group $E_2^{i,j}$ is the i^{th} cohomology of X^n with coefficients in the sheaf on X^n associated to the presheaf $U \mapsto H^j(U \cap \mathcal{F}(X,n);\mathbb{Z})$, ($U \subseteq X^n$ open). The strategy is to exploit the local Euclidean structure of X: Given any point $x \in X^n$, we can associate a partition $\mathscr{P}$ of $[n]$ according to the number of repeated coordinates; with $\overline{\mathscr{P}} = (i_1, \ldots, i_{n-k})$. Then in a sufficiently small neighbourhood U of x,

$$H^j(U \cap \mathcal{F}(X,n);\mathbb{Z}) \cong H^j(\mathcal{F}(\mathbb{R}^m, i_1) \times \cdots \times \mathcal{F}(\mathbb{R}^m, i_{n-k});\mathbb{Z}).$$

In the same manner that Arnol'd computed the cohomology of $\mathcal{F}(\mathbb{C},n)$, we can determine $H^*(\mathcal{F}(\mathbb{R}^m,i);\mathbb{Z})$, a graded-commutative algebra with generators $G_{a,b}$ of degree $(m-1)$, satisfying relations $(0)-(2)$. As in the proof of theorem 4.3, we can decompose $H^*(\mathcal{F}(\mathbb{R}^m, i_1) \times \cdots \times \mathcal{F}(\mathbb{R}^m, i_{n-k});\mathbb{Z})$ into the direct sum of the pullbacks of the top-degree cohomology classes over all refinements of $\mathscr{P}$. We conclude that the non-zero integral groups $E_2^{i,k(m-1)}$ $(0 \leq k \leq n-1)$ are isomorphic to $\bigoplus_{\mathscr{P}} H^i(X_{\mathscr{P}};\mathbb{Z}) \otimes \mathbb{Z}^{c_{\mathscr{P}}}$, the sum over all partitions $\mathscr{P}$ of $[n]$ into $(n-k)$ blocks, where $X_{\mathscr{P}} \cong X^{n-r}$ is the diagonal associated to $\mathscr{P}$, and $c_{\mathscr{P}} = (i_1-1)!\ldots(i_{n-k}-1)!$ is the product of the dimensions of the top degrees. We deduce the ring structure described in the theorem.

If X is a smooth complex projective variety, its rational cohomology has a natural filtration, the weight filtration. Since the map $\mathcal{F}(X,n) \hookrightarrow X^n$ is algebraic, this filtration is compatible with the Leray spectal sequence. This compatibility can be shown by redefining the variety over a suitable subfield $K \subseteq \mathbb{C}$. Then the Galois group Gal$(\overline{K}/K)$ acts on the l-adic cohomology, maps, and spectral sequences, and the weight filtration may be described as a filtration of eigenspaces of a Frobenius element.

The group $E_2^{i,k(m-1)}$ is pure of weight $(i+km)$. Since the differentials respect these weights, we conclude:

Theorem 5.2. *Let X be a smooth complex projective variety. Then d_m is the only possible non-zero differential, and the rational cohomology $H^*(\mathcal{F}(X,n);\mathbb{Q})$ is given by the homology of the algebra E_2 described in theorem 5.1.*

From these theorems, we can further deduce the cohomology of the unordered configuration space $\mathcal{C}(X,n) = \mathcal{F}(X,n)/S_n$: it is then the ring of invariants $H^*(\mathcal{F}(X,n);\mathbb{Q})^{S_n}$ of $H^*(\mathcal{F}(X,n);\mathbb{Q})$ under the induced S_n action. This relationship motivates the study of the cohomology groups $H^*(\mathcal{F}(X,n);\mathbb{Q})$ as S_n-representations, an exciting research area of current interest.

References

[1] V. I. Arnol'd. The cohomology ring of the colored braid group. *Mathematical Notes*, 5(2):138–140, 1969.

[2] T. Church and B. Farb. Representation theory and homological stability. *Arxiv preprint arXiv:1008.1368*, 2010.

[3] B. Farb and D. Margalit. *A Primer on the Mapping Class Group*. 2010. In preparation.

[4] W. Fulton and J. Harris. *Representation Theory: A First Course*. Springer, 2004.

[5] D.J. Hemmer. Stable decompositions for some symmetric group characters arising in braid group cohomology. *Journal of Combinatorial Theory, Series A*, 2010.

[6] B. Totaro. Configuration spaces of algebraic varieties. *Topology*, 35(4):1057–1067, 1996.