Coxeter-Yang-Baxter Equations?

John R. Stembridge

Department of Mathematics
University of Michigan
Ann Arbor, Michigan 48109–1109

February 1993

A Note to the Reader

*These rough notes report on some calculations and observations I made in connection with a study of symmetric functions associated to stable Schubert polynomials and the combinatorics of reduced words. I know that hand-written copies of it did circulate—for example, it is cited in [**1**], but I never planned on publishing it.*

Recently, Fernando Delgado generously volunteered to convert the handwritten notes to TEX, so it is now available to the public. Thanks, Fernando!

John Stembridge, 8 January 2005

Let V be the real Euclidean plane (i.e., $\mathbb{R}^2$), and let R be a rank 2 root system (not necessarily crystallographic) of type $I_2(m)$. Let $r_1, r_2, \ldots, r_m$ denote a system of positive roots for R, ordered as they appear in V, either in clockwise or counter-clockwise order. In this arrangement, r_1 and r_m are the simple roots of the chosen positive system.

Now let $\mathbf{k}$ be a field containing $\mathbb{R}$ and A an associative $\mathbf{k}$-algebra. We can identify the symmetric algebra $S(V)$ with the polynomial ring $\mathbf{k}[x,y]$. In particular, we regard the roots r_i as linear polynomials in $\mathbf{k}[x,y]$. We will say that two elements $g(z), h(z) \in A \otimes \mathbf{k}[z]$ (z an indeterminate) satisfy the *m-th Coxeter-Yang-Baxter Equation* (CYBE) if

$$\underbrace{g(r_1)h(r_2)g(r_3)h(r_4)\cdots}_{m \text{ terms}} = \underbrace{h(r_m)g(r_{m-1})h(r_{m-2})g(r_{m-3})\cdots}_{m \text{ terms}} \qquad (m\text{-CYBE})$$

One should associate g with the simple root r_1, and h with r_m. In this way, if we choose the opposite ordering of the positive roots, then the roles of g and h are switched as well.

Examples.

(1) A root system of type $I_2(3) = A_2$ has positive roots x, y, $x+y$, so the 3rd CYBE (or A_2-CYBE) is the usual YBE:

$$g(x)h(x+y)g(y) = h(y)g(x+y)h(x).$$

(2) The 2nd CYBE is simply $g(x)h(y) = h(y)g(x)$.
(3) A crystallographic root system of type $I_2(4)$ (i.e., B_2) has positive roots x, y, $x+y$, $y-x$, and the two possible orderings are x, $x+y$, y, $y-x$ and its reverse. Hence, the 4th CYBE is

$$g(x)h(x+y)g(y)h(y-x) = h(y-x)g(y)h(x+y)g(x).$$

(4) The positive roots for G_2 may be ordered x, $x+y$, $2x+3y$, $x+2y$, $x+3y$, y. Hence, the G_2-CYBE is

$$h(x)g(x+y)h(2x+3y)g(x+2y)h(x+3y)g(y)$$
$$= g(y)h(x+3y)g(x+2y)h(2x+3y)g(x+y)h(x).$$

This exhausts the crystallographic cases; in all other cases one cannot work in $\mathbb{Z}$.

Remark. If m is odd, every root system of type $I_2(m)$ is conjugate by an orthogonal transformation and a dilation. Furthermore (whether m is even or odd), any system of positive roots is conjugate by a transformation belonging to the group W generated by the corresponding reflections. Note also that the two orderings $r_1, r_2, \dots, r_m$ and $r_m, r_{m-1}, \dots, r_1$ are conjugate by means of the reflection that interchanges r_1 and r_m. From these observations, we conclude that for odd m, the condition that $(g(z), h(z))$ satisfies the m-th CYBE does indeed depend only on m; not on

(1) the particular realization of the root system in V,
(2) the particular choice of positive roots,
(3) the choice of clockwise or counter-clockwise ordering of the positive roots.

In particular, (3) implies that $(h(z), g(z))$ also satisfies the m-th CYBE.

On the other hand, if $m \geq 4$ is even, it is no longer true that every root system of type $I_2(m)$ is conjugate by means of a linear transformation. Indeed, aside from an orthogonal transformation, it may be necessary to independently dilate the "even" roots $r_2, r_4, \dots, r_m$ and the "odd" roots $r_1, r_3, \dots, r_{m-1}$. Thus for even m, the solutions of the m-th CYBE depend on a hidden parameter that measures the ratio of the dilations, and the space of solutions for any two realizations of the root system are related by a map of the form $(g(z), h(z)) \mapsto (g(cz), h(z))$ for some constant c. For example, if $(g(z), h(z))$ is a solution of the B_2-CYBE as in Example (3), then $(h(2z), g(z))$ is also a solution of the B_2-CYBE, but $(h(z), g(z))$ is a solution of a different realization of the 4th CYBE.

Suppose that we have elements $h_1(z), \dots, h_n(z) \in A \otimes \mathbf{k}[z]$, and that $M = [m_{ij}]$ is a Coxeter matrix of size n. (Thus $m_{ij} = m_{ji} \in \{2, 3, \dots, +\infty\}$ for $i \neq j$.) We will say that $h_1(z), \dots, h_n(z)$ form a *Coxeter-Yang-Baxter System* of type M if for all $i \neq j$ such that $m_{ij} < \infty$, we have that $h_i(z)$ and $h_j(z)$ satisfy the m_{ij}-th CYBE.

Now suppose that (W, S) is a Coxeter system of type $M = [m_{ij}]$, and that $S = \{s_1, \ldots, s_n\}$. For the moment, we will assume that W is finite, although later we will explain how this assumption can be removed. Let R be a root system for W, and R^+ a choice of positive roots. For each reduced expression $\mathbf{s} := (s_{i_1}, \ldots, s_{i_l})$, we define the *root sequence* of $\mathbf{s}$ to be

$$\rho := \rho(\mathbf{s}) = \big(e_{i_1}, s_{i_1}(e_{i_2}), s_{i_1}s_{i_2}(e_{i_3}), \ldots, s_{i_1}\cdots s_{i_{l-1}}(e_{i_l})\big)$$

where $e_1, \ldots, e_n$ are the simple roots corresponding to $s_1, \ldots, s_n$. It is easy to show that the root sequence is a linear ordering of the positive roots r such that $w^{-1}r \in R^-$, where $w = s_{i_1}\cdots s_{i_l} \in W$. In particular, ρ is a sequence of l distinct positive roots.

Although we will not need to make use of it here, the following result gives an interesting characterization of root sequences:

THEOREM 1. *Let $\rho = (r_1, \ldots, r_l)$ be a linear ordering of l distinct positive roots. Then ρ is a root sequence iff*

(1) *If r, r' occur in ρ, then so does every root in the positive span of r and r'.*
(2) *If r'' is in the positive span of r and r', and r'' occurs in ρ, then r precedes r'' in ρ, or r' precedes r'' in ρ, but not both.*

Proof. This is an easy induction—a good exercise. □

REMARKS.

(1) A similar result has been given by Kraskiewicz, although it is stated for crystallographic (i.e. Weyl) groups W, and the conditions are slightly different.
(2) It is easy to see that the reduced expression $\mathbf{s}$ is determined by the root sequence $\rho(\mathbf{s})$. Thus the study of reduced expressions is equivalent to the study of root sequences. This is essentially the origin of balanced tableaux.
(3) If W is of rank 2, and $w = w_0$, there are two reduced expressions. The two corresponding root sequences are the clockwise and counter-clockwise orderings of the positive roots that occur in the CYB equations.

Returning to the study of CYB-systems, let us regard the roots r as linear polynomials in $\mathbf{k}[x_1, \ldots, x_n]$ by identifying $x_1, \ldots, x_n$ with some basis of V. (Or alternatively, we can simply work in the symmetric algebra $S(V)$.) Let $\mathbf{s} = (s_{i_1}, \ldots, s_{i_l})$ be a reduced expression for some $w \in W$, and let $\rho(\mathbf{s}) := (r_1, \ldots, r_l)$ be the corresponding root sequence. Let us define the following element of $A \otimes \mathbf{k}[x_1, \ldots, x_n] = A \otimes S(V)$:

$$h(\mathbf{s}) := h_{i_1}(r_1)h_{i_2}(r_2)\cdots h_{i_l}(r_l).$$

THEOREM 2. *If $h_1(z), \ldots, h_n(z)$ is a CYB system, then for all $w \in W$ and all reduced expressions $\mathbf{s}$ for w, $h(\mathbf{s})$ depends only on w, not on $\mathbf{s}$.*

Remark. In case $w = w_0$ and W is of type $I_2(m)$, the condition that the two reduced expressions for w_0 yield the same element of $A \otimes S(V)$ is the m-th CYBE.

Proof. Let $\mathbf{s}'$ be a reduced expression that differs from $\mathbf{s}$ only in positions $p+1, \ldots, p+m$ where in fact

$$s_{i_{p+1}}, s_{i_{p+2}}, \ldots, s_{i_{p+m}} = s_i, s_j, s_i, s_j, \ldots$$
$$s'_{i_{p+1}}, s'_{i_{p+2}}, \ldots, s'_{i_{p+m}} = s_j, s_i, s_j, s_i, \ldots$$

for some i, j, where $m = m_{ij} = \mathrm{order}(s_i s_j)$ in W. It is known that every reduced expression for W can be obtained from any other by a series of tranformations of this type. Therefore, it suffices to show that $h(\mathbf{s}) = h(\mathbf{s}')$, or more specifically, that

$$\underbrace{h_i(r_{p+1})h_j(r_{p+2})\cdots}_{m \text{ terms}} = \underbrace{h_j(r'_{p+1})h_i(r'_{p+2})\cdots}_{m \text{ terms}} \tag{1}$$

where $\rho' = (r'_1, \ldots, r'_l)$ is the root sequence of $\mathbf{s}'$. We will be done if we can show that $(r_{p+1}, \ldots, r_{p+m})$ and $(r'_{p+1}, \ldots, r'_{p+m})$ are the two (distinct) root sequences for the longest element in a root system of type $I_2(m)$, since it will then follow that (1) is an instance of the m-th CYBE.

However, from the definition of root sequence, we find that

$$(r_{p+1}, \ldots, r_{p+m}) = (ue_i, us_ie_j, us_is_je_i, \ldots)$$
$$(r'_{p+1}, \ldots, r'_{p+m}) = (ue_j, us_je_i, us_js_ie_j, \ldots),$$

where $u = s_{i_1} \cdots s_{i_p} \in W$, so from this it is clear that these are indeed the two root sequences for the longest element of the parabolic subgroup $\langle us_iu^{-1}, us_ju^{-1} \rangle$ (see the above remark), relative to the simple system $\{ue_i, ue_j\}$. $\square$

Now we construct an example of a CYB system of type M, for any Coxeter matrix M. Let N be the nil Coxeter algebra of type M. Thus N is the associative $\mathbf{k}$-algebra with generators $u_1, \ldots, u_n$, and defining relations $u_i^2 = 0$ $(1 \le i \le n)$ and $[u_i, u_j]_m = [u_j, u_i]_m$, where $m = m_{ij}$ and $[a, b]_m := abab\cdots$ (m factors).

Theorem 3. *$h_1(z), \ldots, h_N(z) := 1 + zu_1, \ldots, 1 + zu_N$ is a CYB system of type M.*

Proof. For all $i \ne j$, we need to show that the pair $(h_i(z), h_j(z))$ satisfies the m_{ij}-th CYB equation. In other words, we only need to treat the rank two case. Thus suppose $N = \langle u_1, u_2 \rangle$ is the nil Coxeter algebra of type $I_2(m)$. Let $r_1, \ldots, r_m$ be the clockwise ordering of the positive roots of $I_2(m)$. We want to prove the identity

$$(1 + r_1u_1)(1 + r_2u_2)(1 + r_3u_1)\cdots = (1 + r_mu_2)(1 + r_{m-1}u_1)(1 + r_{m-2}u_2)\cdots.$$

Let a_l (respectively b_l) denote the coefficient of $[u_1, u_2]_l$ (respectively $[u_2, u_1]_l$) on the left hand side. Note that the right hand side can be obtained from the left by applying the automorphisms $\varphi : u_1 \rightleftarrows u_2$ and $f : r_i \mapsto r_{m+1-i}$. (Both are involutions.)

Thus we seek to prove that $f(a_l) = b_l$ for $l = 1, 2, \ldots, m$. For each $l \geq 0$, define

$$e_l(x_1, \ldots, x_m) := \sum x_{i_1} \cdots x_{i_l},$$

where the sum ranges over indices $1 \leq i_1 < i_2 < \cdots < i_l \leq m$ such that $i_j = j \bmod 2$.

Note that if $m - l$ is odd, one has

$$e_l(x_1, \ldots, x_m) = e_l(x_1, \ldots, x_{m-1}). \tag{2}$$

One can also easily check that when $m - l$ is even, then

$$e_l(x_1, \ldots, x_m) = e_l(x_m, \ldots, x_1), \tag{3}$$

$$e_l(x_1, \ldots, x_m) = x_1 e_{l-1}(x_2, \ldots, x_m) + e_l(x_3, \ldots, x_m). \tag{4}$$

Also, note that from the definitions, we have

$$a_l = e_l(r_1, \ldots, r_m), \;\; b_l = e_l(r_2, \ldots, r_m).$$

If $m - l$ is odd, then (2) and (3) imply that

$$a_l = e_l(r_1, \ldots, r_{m-1}) \;\text{ and }\; f(a_l) = e_l(r_m, \ldots, r_2) = e_l(r_2, \ldots, r_m) = b_l,$$

as desired. Otherwise, if $m - l$ is even, then we have

$$a_l = f(a_l) = e_l(r_1, \ldots, r_m) \;\text{ and }\; b_l = f(b_l) = e_l(r_2, \ldots, r_{m-1}),$$

and we instead seek to prove that $a_l = b_l$. For this, we claim that the action of W on $a_l - b_l \in S(V)$ is skew-symmetric; i.e., $w(a_l - b_l) = \operatorname{sgn}(w)(a_l - b_l)$ for all $w \in W$. Since all skew-symmetric elements in $S(V)$ are divisible by the "Weyl denominator" $r_1 \cdots r_m$, it follows that $a_l - b_l$ must be, if nonzero, of degree at least m; i.e., $l = m$. But in the case $l = m$, we have $a_m = b_m = r_1 \cdots r_m$, directly from the definition.

Thus, it remains to prove that $a_l - b_l$ is skew-symmetric (for $m - l$ even), i.e., $s_1(a_l - b_l) = -(a_l - b_l)$ and $s_2(a_l - b_l) = -(a_l - b_l)$. First consider the action of s_1. From the geometry of $I_2(m)$, we see that s_1 acts on $r_1, \ldots, r_m$ via $r_1 \to -r_1$, $r_2 \to r_m$, $r_3 \to r_{m-1}, \ldots, r_m \to r_2$. Using (2), (3) and (4), we obtain:

$$\begin{aligned} s_1(a_l - b_l) &= s_1[e_l(r_1, \ldots, r_m) - e_l(r_2, \ldots, r_{m-1})] \\ &= s_1[r_1 e_{l-1}(r_2, \ldots, r_m) + e_l(r_3, \ldots, r_m) - e_l(r_2, \ldots, r_{m-1})] \\ &= -r_1 e_{l-1}(r_m, \ldots, r_2) + e_l(r_{m-1}, \ldots, r_2) - e_l(r_m, \ldots, r_3) \\ &= -r_1 e_{l-1}(r_2, \ldots, r_m) + e_l(r_2, \ldots, r_{m-1}) - e_l(r_3, \ldots, r_m) = -(a_l - b_l). \end{aligned}$$

The fact that $s_2(a_l - b_l) = -(a_l - b_l)$ follows similarly, upon interchanging $r_i \leftrightarrow r_{m+1-i}$. (Note that s_2 acts by $r_m \to -r_m$, $r_1 \to r_{m-1}$, $r_2 \to r_{m-2}, \ldots, r_{m-1} \to r_1$.) $\square$

References

[**1**] S. Billey, Kostant polynomials and the cohomology ring for G/B, *Duke Math. J.* **96** (1999), 205–224.