The Partial Order of Dominant Weights

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Contents

0. Introduction

Throughout this paper, $\Phi \subset \mathbf{R}^n$ shall denote a (reduced) crystallographic root system with positive roots Φ^+, simple roots $\alpha_1, \dots, \alpha_n$, inner product $\langle\, , \rangle$, and Weyl group W. (Standard references are [**B1**] and [**H**].) For each $\alpha \in \Phi$, $\alpha^\vee = 2\alpha/\langle \alpha, \alpha \rangle$ denotes the co-root corresponding to α. We let

$$\Lambda = \{\lambda \in \mathbf{R}^n : \langle \lambda, \alpha^\vee \rangle \in \mathbf{Z} \text{ for all } \alpha \in \Phi\}$$

denote the weight lattice, and $\omega_1, \dots, \omega_n$ the fundamental weights (i.e., $\langle \omega_i, \alpha_j^\vee \rangle = \delta_{ij}$). The set of dominant weights (i.e., the nonnegative integral span of the fundamental weights) is denoted Λ^+.

There is a standard partial ordering $<$ of Λ in which $\mu \le \lambda$ if and only if $\lambda - \mu \in \mathbf{N}\Phi^+$; i.e., $\lambda - \mu$ is a nonnegative integral sum of positive roots. The structure of this partial order is trivial—up to isomorphism, it is the disjoint union of f copies of $\mathbf{Z}^n$ (with the usual product order), where f denotes the index of connection (the index of the root lattice $\mathbf{Z}\Phi$ in Λ). However, a much more subtle partial order is the subposet $(\Lambda^+, <)$ formed by the set of dominant weights. It is this poset that is our object of study.

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The poset $(\Lambda^+, <)$ is of fundamental importance for the representation theory of Lie groups and algebras. To give just one illustration of this, consider a (complex) semisimple Lie algebra $\mathfrak{g}$ with Cartan subalgebra $\mathfrak{h}$ and root system $\Phi \subset \mathfrak{h}^*$. Every finite-dimensional $\mathfrak{g}$-module V has a weight-space decomposition $V = \bigoplus_{\mu \in \Lambda} V_\mu$, where $V_\mu = \{v \in V : hv = \mu(h)v \text{ for all } h \in \mathfrak{h}\}$, and it is well-known that the set of dominant weights μ that occur with positive multiplicity (i.e., $\dim V_\mu > 0$) form an order ideal of $(\Lambda^+, <)$. In particular, if V^λ is the irreducible $\mathfrak{g}$-module of highest weight λ, then (assuming μ is dominant) $\dim(V^\lambda_\mu) > 0$ if and only if $\mu \leq \lambda$.

In fact, our original motivation for studying the partial order of dominant weights arose while developing software for Lie-theoretic computations [**St**].[1] For example, to compute dominant weight multiplicities for V^λ via Freudenthal's algorithm, it can be useful to generate in advance all dominant weights $\mu \leq \lambda$. This led us to the problem of describing explicitly the covering relation of $(\Lambda^+, <)$.

For the root systems of type A, the partial order $(\Lambda^+, <)$ is closely related to the dominance order on the set of partitions of an integer. In the dominance order, one defines $(\beta_1, \beta_2, \dots) \preccurlyeq (\alpha_1, \alpha_2, \dots)$ if $\beta_1 + \cdots + \beta_i \leq \alpha_1 + \cdots + \alpha_i$ for all $i \geq 1$. On the other hand, for the root system $\Phi = A_{n-1}$, the dominant weights Λ^+ can be identified with equivalence classes of partitions with at most n parts, two partitions being equivalent if they differ by a multiple of the n-tuple $(1, \dots, 1)$. With this identification, $\lambda, \mu \in \Lambda^+$ satisfy $\mu \leq \lambda$ if and only if there exist partitions α and β of the same integer, equivalent to μ and λ, such that $\alpha \preccurlyeq \beta$.

We prove several basic theorems about the structure of $(\Lambda^+, <)$, some of which can be viewed as generalizations of well-known properties of the dominance order on partitions. For example, we prove that each component of $(\Lambda^+, <)$ is a lattice (Theorem 1.3), and (assuming Φ is irreducible) these lattices are distributive if and only if Φ is of rank at most 2 (Theorems 3.2 and 3.3). It is interesting that these properties can be attributed to features of the Cartan matrix: the lattice property follows from the fact that the Cartan matrix has at most one positive entry in each column, and distributivity requires at most one negative entry per column.

The main results are in Sections 2 and 4. In Section 2, we give a detailed analysis of the covering relation of $(\Lambda^+, <)$. In particular, we prove that λ covers μ in this ordering only if $\lambda - \mu$ belongs to a distinguished subset of the positive roots (Theorem 2.6). It is surprising that even the fact that $\lambda - \mu$ is necessarily a positive root seems not to have appeared previously in the literature. The analogous result for the dominance order is well-known: α covers β in the dominance order only if β can be obtained from α by decreasing α_i and

[1] See `http://www.math.lsa.umich.edu/~jrs/maple.html`.

increasing α_j for some $i < j$ (i.e., subtracting a type A positive root).

In Section 4 we analyze the Möbius function of $(\Lambda^+, <)$. In particular, we prove that if Φ is irreducible, the Möbius function takes on only the values $0, \pm 1, \pm 2$ (Theorem 4.1), and we determine all component lattices in which the values ± 2 occur. For example, if the diagram of Φ is a path, then only the values $0, \pm 1$ occur, which generalizes Brylawski's result for the dominance order [**Br**]. Our proof technique can be viewed as a root system generalization of Greene's approach to the dominance order [**G**].

Warning.

In this paper, the two notions of lattice (discrete subgroups of real vector spaces and partial orders in which every pair of elements has a least upper bound and greatest lower bound) figure prominently. In some cases, such as the root lattice $\mathbf{Z}\Phi$, these structures are even attached to the same object. Nevertheless, it should not be difficult for the reader to discern the meaning of each use of the word "lattice" from its context.

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I would like to thank James Humphreys and Andrey Zelevinsky for helpful discussions.

1. Basic Properties

Let $\Lambda_1, \ldots, \Lambda_f$ denote the distinct cosets of Λ modulo $\mathbf{Z}\Phi$, and let $\Lambda_i^+ = \Lambda^+ \cap \Lambda_i$. It is clear from the definition that $\mu, \nu \in \Lambda$ can be related by $<$ only if they belong to the same coset, so $(\Lambda^+, <)$ is the disjoint union of the subposets $(\Lambda_i^+, <)$.

It should also be noted that if Φ has two or more irreducible factors, then $(\Lambda^+, <)$ is isomorphic to the direct product of the posets corresponding to these factors. In some cases, it will be simpler to restrict our attention to the case of irreducible Φ; extending to the general case is straightforward.

1.1 The Lattice Property.

Lemma 1.1. *Each component $(\Lambda_i^+, <)$ is directed; i.e., every pair $\mu, \nu \in \Lambda_i^+$ has an upper bound.*

Proof. Let $\delta = 2(\omega_1 + \cdots + \omega_n) \in \Lambda^+$. It is well-known that $\delta \in \mathbf{N}\Phi^+$. In fact, δ is the sum of the positive roots (e.g., [**H**, §13.3]), so the simple root coordinates of δ are positive. It follows that if λ is an arbitrary representative of the coset Λ_i, then every pair $\mu, \nu \in \Lambda_i^+$ has an upper bound of the form $\lambda + k\delta$ for k sufficiently large. $\square$

Each component of $(\Lambda, <)$ is isomorphic to a direct product of n copies of $\mathbf{Z}$, and is therefore a lattice. Furthermore, the meet and join operations can be expressed in terms of the simple root coordinates as follows:

$$(\sum_i a_i\alpha_i) \wedge (\sum_i b_i\alpha_i) = \sum_i \min(a_i, b_i)\alpha_i, \tag{1.1}$$

$$(\sum_i a_i\alpha_i) \vee (\sum_i b_i\alpha_i) = \sum_i \max(a_i, b_i)\alpha_i. \tag{1.2}$$

Note that a_i and b_i need not be integers. However, the operands must belong to the same coset, so we have $a_i - b_i \in \mathbf{Z}$ and therefore $a_i - \min(a_i, b_i) \in \mathbf{Z}$. Hence, the above expression for the meet (and similarly the join) does belong to the proper coset.

Lemma 1.2. *Let $\gamma \in \mathbf{Z}\Phi^\vee$. We have*

$$\langle \mu, \gamma\rangle, \langle \nu, \gamma\rangle \geq 0 \ \Rightarrow\ \langle \mu \wedge \nu, \gamma\rangle \geq 0$$

for all μ, ν in the same coset of Λ if and only if there is at most one i such that $\langle \alpha_i, \gamma\rangle > 0$.

Proof. If $\langle \alpha_1, \gamma\rangle = c_1 > 0$ and $\langle \alpha_2, \gamma\rangle = c_2 > 0$, then take $\mu = c_1\alpha_2 - c_2\alpha_1$ and $\nu = 0$. Under these conditions, we have $\mu \wedge \nu = -c_2\alpha_1$, $\langle \mu \wedge \nu, \gamma\rangle = -c_1c_2 < 0$, and $\langle \mu, \gamma\rangle = \langle \nu, \gamma\rangle = 0$, so the stated condition is clearly necessary.

For the converse, suppose $\langle \alpha_1, \gamma\rangle = c_1 \geq 0$ and $\langle \alpha_i, \gamma\rangle = -c_i \leq 0$ for $2 \leq i \leq n$. Given $\mu = \sum_i a_i\alpha_i$ and $\nu = \sum_i b_i\alpha_i$, the condition $\langle \mu, \gamma\rangle, \langle \nu, \gamma\rangle \geq 0$ implies

$$c_1a_1 \geq c_2a_2 + \cdots + c_na_n \geq c_2 \min(a_2, b_2) + \cdots + c_n \min(a_n, b_n),$$

$$c_1b_1 \geq c_2b_2 + \cdots + c_nb_n \geq c_2 \min(a_2, b_2) + \cdots + c_n \min(a_n, b_n),$$

and therefore

$$c_1 \min(a_1, b_1) \geq c_2 \min(a_2, b_2) + \cdots + c_n \min(a_n, b_n).$$

That is, $\langle \mu \wedge \nu, \gamma\rangle \geq 0$. □

Theorem 1.3. *Each component $(\Lambda_i^+, <)$ is (a) a complete meet-semilattice, and (b) a lattice. Furthermore, the meet operation of $(\Lambda_i^+, <)$ is given by (1.1).*

Proof. We first prove that $(\Lambda_i^+, <)$ is a meet-semilattice. For this it suffices to show that $\mu, \nu \in \Lambda_i^+$ implies $\mu \wedge \nu \in \Lambda^+$, where $\wedge$ is defined as in (1.1). Indeed, it is well-known that $\langle \alpha_k, \alpha_j^\vee\rangle \leq 0$ for all $k \neq j$ (e.g., [**H**, §10.1]), so $\gamma = \alpha_j^\vee$ satisfies the hypothesis of Lemma 1.2. This allows us to deduce $\langle \mu \wedge \nu, \alpha_j^\vee\rangle \geq 0$ from the fact that $\langle \mu, \alpha_j^\vee\rangle \geq 0$ and $\langle \nu, \alpha_j^\vee\rangle \geq 0$. In other words, $\mu \wedge \nu$ is dominant, which proves the claim.

The meet of an arbitrary subset of Λ_i^+ can be therefore be expressed in the form $\mu_1 \wedge \cdots \wedge \mu_n$, where μ_j is a member of the subset that minimizes the coefficient of α_j. Thus $(\Lambda_i^+, <)$ is complete as a meet-semilattice. Since $(\Lambda_i^+, <)$ is also directed (Lemma 1.1), it is therefore a lattice. □

Corollary 1.4. *Each component of* $(\Lambda^+, <)$ *has a minimum element.*

Remark 1.5. (a) The above argument shows that the lattice property depends ultimately on the fact that the Cartan matrix $[\langle \alpha_i, \alpha_j^\vee \rangle]$ has (at most) one positive entry in each column.

(b) In Section 3, we shall see that the join operation of $(\Lambda_i^+, <)$ is not necessarily given by (1.2), so $(\Lambda_i^+, <)$ need not be a sublattice of $(\Lambda_i, <)$.

(c) In the case $\Phi = A_{n-1}$, the dominance ordering of partitions of n is isomorphic to a subinterval of $(\Lambda^+, <)$. Hence a corollary of Theorem 1.3 is the well-known fact that the dominance order is a lattice (e.g., see [**Br**]).

1.2 Saturation.

The material in this subsection is not new—it is based on the exercises in Bourbaki (see especially Exercises VI.1.23–24 and VI.2.5 of [**B1**]).

A subset Σ of Λ is said to be *saturated* if for every $\lambda \in \Sigma$, $\alpha \in \Phi$, and integer i satisfying $0 < i \leq \langle \lambda, \alpha^\vee \rangle$, we have $\lambda - i\alpha \in \Sigma$.

Lemma 1.6. *Saturated sets are* W*-stable.*

Proof. Assume $\Sigma \subset \Lambda$ is saturated, $\lambda \in \Sigma$ and $\alpha \in \Phi$. We must have $\lambda - \langle \lambda, \alpha^\vee \rangle \alpha \in \Sigma$, since even if $\langle \lambda, \alpha^\vee \rangle < 0$, we can replace α with $-\alpha$. However, $\lambda - \langle \lambda, \alpha^\vee \rangle \alpha$ is the reflection of λ through the hyperplane orthogonal to α. Since W is generated by such reflections, the result follows. $\square$

Lemma 1.7. *If* $\mu \in \Lambda^+$ *and* $w \in W$*, then* $w\mu \leq \mu$*.*

Proof. As a subposet of $(\Lambda, <)$, the W-orbit of μ has at least one maximal element, say μ^+. However μ^+ must be dominant, since $\langle \mu^+, \alpha_i^\vee \rangle = -c < 0$ would imply that the reflection of μ^+ through the hyperplane orthogonal to α_i is $\mu^+ + c\alpha_i > \mu^+$, a contradiction. Since each W-orbit has just one dominant vector, it follows that $\mu = \mu^+$ is the unique maximal element of its orbit. $\square$

Lemma 1.8. *If* Σ *is saturated,* $\lambda \in \Sigma$*, and* $\mu \in \Lambda^+$*, then* $\mu < \lambda$ *implies* $\mu \in \Sigma$*.*

Proof. If not, then there must exist $\nu \in \Sigma$ satisfying $\mu < \nu \leq \lambda$, but with $\nu - \alpha_i \notin \Sigma$ for all simple roots α_i in the support of $\nu - \mu$. Setting $\nu - \mu = \sum_{i \in I} b_i \alpha_i$ with $b_i > 0$, we have $\sum_{i \in I} b_i \langle \nu - \mu, \alpha_i \rangle = \langle \nu - \mu, \nu - \mu \rangle > 0$, so $\langle \nu - \mu, \alpha_i^\vee \rangle > 0$ for some $i \in I$. Furthermore, μ is dominant by hypothesis, so it must be the case that $\langle \nu, \alpha_i^\vee \rangle > 0$. However Σ is saturated, so we must have $\nu - \alpha_i \in \Sigma$, a contradiction. $\square$

For $\lambda \in \Lambda^+$, define $\Sigma(\lambda)$ to be the smallest saturated subset of Λ that contains λ. (Since intersections of saturated sets are saturated, it is clear that a smallest saturated subset exists.) A weaker version of the following result, corresponding to the inclusion '$\subseteq$' is the main point of Exercise VI.1.23 of [**B1**]; the reverse inclusion does not seem to be stated explicitly anywhere in [**B1**], but it is implicit in [**B2**, VIII.7.2].

THEOREM 1.9. *For $\lambda \in \Lambda^+$, we have*

$$\Sigma(\lambda) = \{\mu \in \Lambda : w\mu \leq \lambda \textit{ for all } w \in W\} = \bigcup_{\mu \in \Lambda^+ :\, \mu \leq \lambda} W\mu.$$

Proof. The equality of the second and third expressions is a consequence of Lemma 1.7. Also, Lemmas 1.6 and 1.8 imply that $\Sigma(\lambda) \supseteq W\mu$ for all $\mu \in \Lambda^+$ such that $\mu \leq \lambda$. Hence all that remains is to show that $\bar{\Sigma}(\lambda) := \{\mu \in \Lambda : w\mu \leq \lambda \text{ for all } w \in W\}$ is saturated.

Thus suppose $\mu \in \bar{\Sigma}(\lambda)$ and $\alpha \in \Phi$. Given $0 < i \leq \langle \mu, \alpha^\vee \rangle$ and $w \in W$, consider $w(\mu - i\alpha)$. If $w\alpha \in \Phi^+$, then we have

$$\lambda \geq w\mu \geq w\mu - i(w\alpha) = w(\mu - i\alpha),$$

whereas if $-w\alpha \in \Phi^+$, then

$$\lambda \geq wt_\alpha\mu = w(\mu - \langle \mu, \alpha^\vee \rangle \alpha) = w\mu - \langle \mu, \alpha^\vee \rangle w\alpha \geq w\mu - iw\alpha = w(\mu - i\alpha),$$

where $t_\alpha \in W$ denotes the reflection through the hyperplane orthogonal to α. We therefore have $w(\mu - i\alpha) \leq \lambda$ for all $w \in W$, so $\mu - i\alpha \in \bar{\Sigma}(\lambda)$ and $\bar{\Sigma}(\lambda)$ is saturated. □

COROLLARY 1.10. *For $\lambda, \mu \in \Lambda^+$, we have $\mu \leq \lambda$ if and only if $\Sigma(\mu) \subseteq \Sigma(\lambda)$.*

REMARK 1.11. If V^λ is the irreducible $\mathfrak{g}$-module of highest weight λ, then it follows from Proposition 5 of [**B2**, VIII.7.2] that $\Sigma(\lambda)$ is the set of weights that occur with nonzero multiplicity in V^λ. Along with the above corollary, this proves the assertion mentioned in the introduction; namely, that for $\mu \in \Lambda^+$, $\dim(V^\lambda_\mu) > 0$ if and only if $\mu \leq \lambda$.

A dominant weight λ is *minuscule* if it is nonzero and $\langle \lambda, \alpha^\vee \rangle \in \{0, \pm 1\}$ for all $\alpha \in \Phi$.

PROPOSITION 1.12. *A dominant weight λ is a minimal element of $(\Lambda^+, <)$ if and only if $\lambda = 0$ or λ is minuscule.*

Proof. If λ is minuscule (or zero), then for any $\alpha \in \Phi$, $0 < i \leq \langle \lambda, \alpha^\vee \rangle$ can occur only if $\langle \lambda, \alpha^\vee \rangle = i = 1$. In that case, $\lambda - i\alpha$ is $t_\alpha\lambda$, the reflection of λ through the hyperplane orthogonal to α. Hence $W\lambda$ is itself saturated, and therefore $\Sigma(\lambda) = W\lambda$. By Theorem 1.9, it follows that λ is a minimal element of $(\Lambda^+, <)$.

Conversely, if λ is nonzero and not minuscule, then there must be a root α such that $\langle \lambda, \alpha^\vee \rangle \geq 2$. In that case, $\lambda - \alpha$ is an interior point of the line segment from λ to $t_\alpha\lambda$.

However t_α is an isometry, so λ and $t_\alpha\lambda$ are at the same distance from the origin. Hence $\lambda - \alpha$ must be strictly closer to zero; in particular, it cannot belong to the W-orbit of λ. Thus $\Sigma(\lambda)$, which necessarily contains $\lambda - \alpha$ (by saturation) has more than one W-orbit, whence by Theorem 1.9, λ cannot be a minimal element of $(\Lambda^+, <)$. $\square$

Combining Corollary 1.4 and the above result we obtain the following.

COROLLARY 1.13. *Each nontrivial coset of Λ contains exactly one minuscule weight. In particular, the number of minuscule weights is $f - 1$.*

2. The Covering Relation

Assume temporarily that Φ is irreducible. In that case, the roots form either one or two orbits according to whether Φ is simply or multiply-laced. In the latter case, the roots in the two orbits have different lengths, "long" and "short," and the squared ratio of these lengths is either two or three. (See [**H**, §10.4], for example.) In the simply-laced case, it is convenient to say that the roots are both long and short. With this convention, Φ has exactly one long root that is dominant (the so-called *highest root*), and one short dominant root. The latter will be denoted $\bar{\alpha}$.

The following result is equivalent to Exercise VIII.7.22 of [**B2**].

PROPOSITION 2.1. *If Φ is irreducible and $\lambda \in \Lambda^+$, then $\lambda > 0$ implies $\lambda \geq \bar{\alpha}$.*

Proof. Choose a nonzero dominant $\mu \leq \lambda$ of minimum length. The weight μ cannot be minuscule (Proposition 1.12), so there is a root α such that $\langle \mu, \alpha^\vee \rangle \geq 2$. By reasoning similar to the proof of Proposition 1.12, it follows that $\mu - \alpha \in \Sigma(\mu) \subseteq \Sigma(\lambda)$ is shorter than μ, which contradicts the choice of μ unless $\mu - \alpha = 0$. That is, $\mu = \alpha$ is a (dominant) root. It must also be the case that α is short, since the long dominant root is the unique maximal element of $(\Phi, <)$ (e.g., Proposition VI.1.25 of [**B1**]). $\square$

REMARK 2.2. A dominant weight λ is said to be *quasi-minuscule* if λ covers 0 in $(\Lambda^+, <)$. By Theorem 1.9, this is equivalent to $\Sigma(\lambda) = W\lambda \,\dot\cup\, \{0\}$. The above result shows that in the irreducible case there is exactly one quasi-minuscule weight: $\bar{\alpha}$.

LEMMA 2.3. *For $\alpha, \beta \in \Phi$, we have $\langle \alpha, \beta^\vee \rangle \in \{0, \pm 1\}$ unless $\alpha = \pm\beta$ or α is (strictly) longer than β.*

Proof. Suppose $\langle \alpha, \beta^\vee \rangle \geq 2$. We have $\langle \alpha, \beta^\vee \rangle \langle \beta, \alpha^\vee \rangle = 4\cos^2\theta \leq 4$, where θ denotes the angle between α and β, so $\langle \beta, \alpha^\vee \rangle \leq 2$. Hence $\langle \alpha, \alpha \rangle / \langle \beta, \beta \rangle = \langle \alpha, \beta^\vee \rangle / \langle \beta, \alpha^\vee \rangle \geq 1$, so either α is longer than β, or they have the same length and $\cos^2\theta = 1$; i.e. $\alpha = \pm\beta$. $\square$

For $\beta = \sum_i b_i \alpha_i \in \mathbf{Z}\Phi$, let $\operatorname{Supp} \beta = \{i : b_i \neq 0\}$.

Lemma 2.4. *If $\lambda \in \Lambda^+$, $\beta \in \mathbf{N}\Phi^+$, and $\langle \lambda - \beta, \alpha_i^\vee \rangle \geq 0$ for all $i \in \operatorname{Supp}\beta$, then $\lambda - \beta$ is dominant.*

Proof. Recall that $\langle \alpha_j, \alpha_i^\vee \rangle \leq 0$ for $i \neq j$. It follows that $\langle \beta, \alpha_i^\vee \rangle \leq 0$ for all $i \notin \operatorname{Supp}\beta$, and hence $\langle \lambda - \beta, \alpha_i^\vee \rangle \geq 0$. □

For $I \subseteq \{1, \dots, n\}$, let Φ_I denote the root subsystem generated by $\{\alpha_i : i \in I\}$. If Φ_I is irreducible, we let $\bar{\alpha}_I$ denote the short dominant root of Φ_I. We say that $\bar{\alpha}_I$ is a *locally short dominant root* of Φ; the modification "local" applies to both length and dominance, since $\bar{\alpha}_I$ may be long in Φ but short in Φ_I.

For $\beta = \sum_i b_i \alpha_i \in \mathbf{Z}\Phi$ and $I \subseteq \{1, \dots, n\}$, let $\beta|_I = \sum_{i \in I} b_i \alpha_i$.

Lemma 2.5. *Suppose $\mu < \mu + \beta$ in $(\Lambda^+, <)$, $I = \operatorname{Supp}\beta$, $J = \{i \in I : \langle \mu, \alpha_i^\vee \rangle = 0\}$, and that Φ_K is an irreducible subsystem of Φ_I ($K \subseteq I$).*

(a) *If $\langle \beta|_K, \alpha_i^\vee \rangle \geq 0$ for all $i \in K - J$, then $\beta \geq \bar{\alpha}_K$.*

(b) *If in addition, $\langle \mu + \bar{\alpha}_K, \alpha_i^\vee \rangle \geq 0$ for all $i \in I - K$, then $\mu + \bar{\alpha}_K$ is dominant.*

Proof. (a) For $i \in J$, we have $\langle \beta, \alpha_i^\vee \rangle = \langle \mu + \beta, \alpha_i^\vee \rangle \geq 0$, since $\mu + \beta$ is dominant. It follows that if $i \in K \cap J$, then

$$\langle \beta|_K, \alpha_i^\vee \rangle = \langle \beta, \alpha_i^\vee \rangle - \langle \beta - \beta|_K, \alpha_i^\vee \rangle \geq \langle \beta, \alpha_i^\vee \rangle \geq 0,$$

since i is not in the support of $\beta - \beta|_K$. Combining this with the stated hypothesis, we obtain $\langle \beta|_K, \alpha_i^\vee \rangle \geq 0$ for all $i \in K$, so $\beta \geq \beta|_K \geq \bar{\alpha}_K$ by Proposition 2.1.

(b) Given (a), we have that $\beta - \bar{\alpha}_K \geq 0$. Setting $\lambda = \mu + \beta$ (a dominant weight by hypothesis), we have $\mu + \bar{\alpha}_K = \lambda - (\beta - \bar{\alpha}_K)$. By Lemma 2.4, it suffices to prove that $\langle \mu + \bar{\alpha}_K, \alpha_i^\vee \rangle \geq 0$ for all $i \in I$. For $i \in I - K$ this is part of the stated hypothesis, so we need only to prove it for $i \in K$. However $\bar{\alpha}_K$ is dominant relative to Φ_K, so $\langle \bar{\alpha}_K, \alpha_i^\vee \rangle \geq 0$ for $i \in K$ and the claim follows. □

Theorem 2.6. *If λ covers μ in $(\Lambda^+, <)$ and $I = \operatorname{Supp}(\lambda - \mu)$, then either $\lambda - \mu = \bar{\alpha}_I$, or $\Phi_I \cong G_2$ and $\lambda - \mu = \sum_{i \in I} \alpha_i$.*

Proof. Let $\beta = \lambda - \mu$, $I = \operatorname{Supp}\beta$, and $J = \{i \in I : \langle \mu, \alpha_i^\vee \rangle = 0\}$, as in the statement of Lemma 2.5. It suffices to identify some $K \subseteq I$ meeting the hypotheses of Lemma 2.5, since in that case we deduce that $\mu + \bar{\alpha}_K$ is dominant and $\mu < \mu + \bar{\alpha}_K \leq \lambda$ (since $\bar{\alpha}_K \leq \beta$). However λ is assumed to cover μ, so this is possible only if $\lambda = \mu + \bar{\alpha}_K$ and $K = I$.

Case I: J is empty. In this case let $K = \{i\}$, where $i \in I$ is chosen so that α_i is short relative to Φ_I. We have $\beta|_K = b\alpha_i$ for some $b \geq 1$, so the hypothesis (and conclusion) of Lemma 2.5(a) is trivial. Since α_i is short, we have $\langle \alpha_i, \alpha_j^\vee \rangle \geq -1$ (Lemma 2.3) and

$\langle \mu, \alpha_j^\vee \rangle \geq 1$ (J is empty) for all $j \in I$. Hence $\langle \mu + \alpha_i, \alpha_j^\vee \rangle \geq 0$ for all $i \in I$, and the hypotheses of Lemma 2.5 are satisfied.

We assume henceforth that J is nonempty. Choose $K \subseteq J$ so that Φ_K is an irreducible component of Φ_J containing a root that is short relative to Φ_J. It follows in particular that $\bar{\alpha}_K$ must be short relative to Φ_J.

Case II: $\langle \mu + \bar{\alpha}_K, \alpha_i^\vee \rangle \geq 0$ for all $i \in I - J$. The hypothesis of Lemma 2.5(a) is vacuous in this case, since $K \subseteq J$. Also, since Φ_K is an irreducible component of Φ_J, we have $\langle \bar{\alpha}_K, \alpha_i^\vee \rangle = 0$ for all $i \in J - K$, and hence $\langle \mu + \bar{\alpha}_K, \alpha_i^\vee \rangle \geq 0$ for $i \in J - K$. Combining this with the stated premise for this case yields the hypothesis for Lemma 2.5(b).

We may assume henceforth that there is some $i \in I - J$ for which $\langle \mu + \bar{\alpha}_K, \alpha_i^\vee \rangle < 0$. Since $i \notin J$ implies $\langle \mu, \alpha_i^\vee \rangle \geq 1$, this is possible only if

$$\langle \bar{\alpha}_K, \alpha_i^\vee \rangle \leq -2. \tag{2.1}$$

Now choose $L \subseteq I$ so that Φ_L is the irreducible component of $\Phi_{J \cup \{i\}}$ that contains α_i. Note that (2.1) implies $K \subset L$ and that α_i is strictly shorter than $\bar{\alpha}_K$ (Lemma 2.3). In particular, Φ_L is multiply-laced.

Case III: $\langle \bar{\alpha}_K, \alpha_i^\vee \rangle = -2$. In this case, the square of the length ratio of long and short roots must be 2. Furthermore, since $\bar{\alpha}_K$ is long relative to Φ_L but short (by choice) relative to Φ_J, it must be the case that *every* simple root of Φ_L other than α_i is long. Hence $\langle \alpha_j, \alpha_i^\vee \rangle \in \{0, -2\}$ for all $j \in L - \{i\}$. Since $\langle \alpha_i, \alpha_i^\vee \rangle = 2$, it follows that

$$\langle \gamma, \alpha_i^\vee \rangle \text{ is even for all } \gamma \in \mathbf{Z}\Phi_L. \tag{2.2}$$

Now since $\langle \mu + \bar{\alpha}_K, \alpha_i^\vee \rangle = \langle \mu, \alpha_i^\vee \rangle - 2 < 0$ and $\langle \mu, \alpha_i^\vee \rangle \geq 1$, it must be the case that $\langle \mu, \alpha_i^\vee \rangle = 1$. Also, since $i \notin \mathrm{Supp}(\beta - \beta|_L)$, we have $\langle \beta - \beta|_L, \alpha_i^\vee \rangle \leq 0$, and hence

$$\langle \beta|_L, \alpha_i^\vee \rangle = \langle \beta, \alpha_i^\vee \rangle - \langle \beta - \beta|_L, \alpha_i^\vee \rangle \geq \langle \beta, \alpha_i^\vee \rangle = \langle \lambda, \alpha_i^\vee \rangle - \langle \mu, \alpha_i^\vee \rangle \geq -1.$$

However $\langle \beta|_L, \alpha_i^\vee \rangle$ must be even by (2.2), so $\langle \beta|_L, \alpha_i^\vee \rangle \geq 0$.

Using L in the role of K, the above argument proves that the hypothesis of Lemma 2.5(a) holds (since $L - J = \{i\}$). Furthermore, since α_i is strictly shorter than $\bar{\alpha}_K$, it is also short relative to the irreducible component of Φ_I that contains it, and hence the same is true for $\bar{\alpha}_L$. Therefore $\langle \bar{\alpha}_L, \alpha_j^\vee \rangle \geq -1$ for all $j \in I$ (Lemma 2.3). Since $\langle \bar{\alpha}_L, \alpha_j^\vee \rangle = 0$ for all $j \in J - L$ and $\langle \mu, \alpha_j^\vee \rangle \geq 1$ for all $j \notin J$, it follows that $\langle \mu + \bar{\alpha}_L, \alpha_j^\vee \rangle \geq 0$ for all $j \in I - L$, and hence the hypothesis of Lemma 2.5(b) (with $K = L$) holds.

Case IV: $\langle \bar{\alpha}_K, \alpha_i^\vee \rangle = -3$. In this case, α_i and $\bar{\alpha}_K$ generate a root subsystem isomorphic to G_2. Since G_2 is the only irreducible root system that contains G_2, this can happen only if $\bar{\alpha}_K$ is a simple root, say α_j, and $L = \{i, j\}$. Now since $\langle \mu + \bar{\alpha}_K, \alpha_i^\vee \rangle < 0$, $\langle \mu, \alpha_i^\vee \rangle \geq 1$, and $\langle \bar{\alpha}_K, \alpha_i^\vee \rangle = -3$, we have $\langle \mu, \alpha_i^\vee \rangle \in \{1, 2\}$. Hence

$$\langle \mu + \alpha_i + \alpha_j, \alpha_i^\vee \rangle \geq 1 + 2 - 3 \geq 0,$$
$$\langle \mu + \alpha_i + \alpha_j, \alpha_j^\vee \rangle \geq 0 - 1 + 2 \geq 1,$$

and since $\alpha_i + \alpha_j$ is orthogonal to all remaining simple roots, it follows that $\mu + \alpha_i + \alpha_j$ is dominant. It is also clear that $\mu + \alpha_i + \alpha_j \leq \lambda$ since $\{i, j\} \subseteq I = \mathrm{Supp}(\lambda - \mu)$. However λ covers μ, so this is possible only if $\lambda = \mu + \alpha_i + \alpha_j$ and $I = \{i, j\}$. □

Since the sum of the two simple roots of G_2 is a root, we obtain the following.

Corollary 2.7. *If λ covers μ in $(\Lambda^+, <)$, then $\lambda - \mu \in \Phi^+$.*

So far as we have been able to determine, the above Corollary is new, or at least not easily found in the literature. However the following elegant proof, independent of Theorem 2.6, was recently obtained by Robert Steinberg and communicated to us by James Humphreys.

Second Proof of Corollary 2.7. Suppose λ and μ are dominant weights satisfying $\lambda > \mu$. Among all expressions $\lambda - \mu = \beta_1 + \cdots + \beta_l$ with $\beta_i \in \Phi^+$, choose one that maximizes the sum of the simple root coordinates of β_1. If $\mu + \beta_1$ were not dominant, say $\langle \mu + \beta_1, \alpha_i^\vee \rangle < 0$, then we would have $\langle \beta_1, \alpha_i^\vee \rangle < 0$, so $\beta_1 + \alpha_i$ would be a (positive) root. Moreover, since $\lambda = (\mu + \beta_1) + (\beta_2 + \cdots + \beta_l)$ is dominant, we must also have $\langle \beta_2 + \cdots + \beta_l, \alpha_i^\vee \rangle > 0$. Reordering indices if necessary, we may assume that $\langle \beta_2, \alpha_i^\vee \rangle > 0$. But then $\beta_2 - \alpha_i$ is a positive root or zero, and the expression

$$\lambda - \mu = (\beta_1 + \alpha_i) + (\beta_2 - \alpha_i) + \beta_3 + \cdots + \beta_l$$

contradicts the choice of β_1. Therefore $\mu + \beta_1$ is dominant and $\mu < \mu + \beta_1 \leq \lambda$. Given that λ covers μ, this implies $\lambda - \mu = \beta_1$. □

It will be convenient to say that a root $\alpha \in \Phi$ is *exceptional* if it is the sum of two simple roots of Φ that generate a root system isomorphic to G_2.

Let $E(\Phi)$ denote the set of roots appearing Theorem 2.6; i.e., the set of locally short dominant roots of Φ, together with the exceptional roots. It follows from the above theorem that these roots generate $(\Lambda^+, <)$ in the sense that the partial order is the transitive closure of all relations $\mu < \mu + \alpha$ with $\alpha \in E(\Phi)$. Of course, not all relations of this form are covering relations. The following strengthening of Theorem 2.6 clarifies this precisely.

THEOREM 2.8. *If $\lambda > \mu$ in $(\Lambda^+, <)$, $I = \mathrm{Supp}(\lambda - \mu)$, and $J = \{i \in I : \langle \mu, \alpha_i^\vee \rangle = 0\}$, then λ covers μ if and only if Φ_I is irreducible and one of the following holds:*

(a) *$\lambda - \mu$ is a simple root.*
(b) *$I = J$ and $\lambda - \mu = \bar{\alpha}_I$.*
(c) *$I = J \cup \{i\}$, Φ_I is of type B, α_i is short, $\langle \mu, \alpha_i^\vee \rangle = 1$, and $\lambda - \mu = \bar{\alpha}_I$.*
(d) *$I = J \cup \{i\}$, $\Phi_I \cong G_2$, α_i is short, $\langle \mu, \alpha_i^\vee \rangle \in \{1, 2\}$, and $\lambda - \mu \in \Phi_I$ is exceptional.*

Proof. If λ covers μ, then one of the Cases I–IV identified in the proof of Theorem 2.6 must apply. In fact, Cases I, II, III, and IV give rise to configurations of the type described in (a), (b), (c), and (d), respectively. (One should note that in Case III, Φ_I has only one short simple root, and that the squared ratio of root lengths is two. These circumstances alone are sufficient to imply that Φ_I must be of type B.)

It therefore suffices to show that each of the configurations described above is in fact a covering relation. For (a) this is clear. In the remaining cases, we have $\lambda - \mu \in E(\Phi)$; say $\lambda - \mu = \alpha$. If $\mu + \alpha$ failed to cover μ, then by Theorem 2.6 there would exist some $\beta < \alpha$ in $E(\Phi)$ such that $\mu + \beta$ is dominant. However $\beta < \alpha$ implies that β is a non-dominant root of Φ_I. Hence there must be some $i \in I$ such that $\langle \beta, \alpha_i^\vee \rangle < 0$, which contradicts the fact that $\mu + \beta$ is dominant unless $i \notin J$. For (b) there is nothing further to prove, but in (c) and (d) we still have the possibility that α_i is short and that Φ_I is of type B or G_2. The Cartan integers of these root systems are such that $\langle \beta, \alpha_i^\vee \rangle < 0$ implies $\langle \beta, \alpha_i^\vee \rangle = -2$ (in type B) or $\langle \beta, \alpha_i^\vee \rangle = -3$ (in G_2), which for dominance of $\mu + \beta$ requires $\langle \mu, \alpha_i^\vee \rangle \geq 2$ and $\langle \mu, \alpha_i^\vee \rangle \geq 3$ respectively, a contradiction. □

REMARK 2.9. This result shows that $E(\Phi)$ is the minimum generating set for $(\Lambda^+, <)$; that is, for each $\alpha \in E(\Phi)$, there exists a covering pair $\lambda > \mu$ in Λ^+ such that $\lambda - \mu = \alpha$. In fact, suppose that $\alpha = \bar{\alpha}_I$ is a locally short dominant root and $\mu = \sum_{i \notin I} m_i \omega_i$. Since α is locally dominant, it follows that $\mu + \alpha$ is dominant if the m_i's are sufficiently large, and Theorem 2.8 then implies that $\mu + \alpha$ covers μ. (If $\alpha = \alpha_1 + \alpha_2$ is exceptional and α_1 is short, take $\mu = \omega_1 + \sum_{i > 2} m_i \omega_i$.) This shows furthermore that each $\alpha \in E(\Phi)$ occurs infinitely often as the difference between covering pairs in $(\Lambda^+, <)$, except possibly if Φ is irreducible and $\alpha = \bar{\alpha}$ or α is exceptional. In these cases, Theorem 2.8 shows that $\mu + \bar{\alpha}$ covers μ if and only if $\mu = 0$ (cf. Proposition 2.1), or μ is minuscule and $\Phi \cong A_1$ or B_n. If $\Phi = G_2$ and α is exceptional, then $\mu + \alpha$ covers μ if and only if $\mu = \bar{\alpha}$ or $2\bar{\alpha}$.

Define $E^*(\Phi)$ to be the set consisting of those roots $\alpha \in E(\Phi)$ such that Φ_I is not isomorphic to a root system of type A, where $I = \mathrm{Supp}\,\alpha$. For such α we claim that there is a unique index $p = p(\alpha) \in I$ such that $\langle \alpha, \alpha_p^\vee \rangle > 0$. If α is exceptional, this is an easy calculation (in fact p is the index of the long simple root), whereas if α is a locally short dominant root, this follows from the familiar fact that the extended diagram of a root system not of type A is acyclic.

Proposition 2.10. *The map $\alpha \mapsto p(\alpha)$ is a bijection $E^*(\Phi) \to \{i : \omega_i$ not minuscule$\}$.*

Proof. Let $\alpha \in E^*(\Phi)$, $I = \operatorname{Supp} \alpha$, and $p = p(\alpha)$. Since α is short relative to Φ_I and $\alpha \neq \alpha_p$ (otherwise $\Phi_I \cong A_1$), Lemma 2.3 implies $\langle \alpha, \alpha_p^\vee \rangle = 1$. Also, since $\langle \alpha, \alpha_i^\vee \rangle = 0$ for all $i \in I$ except $i = p$, it follows that $\omega_p - \alpha$ is dominant (Lemma 2.4). Therefore ω_p is not a minimal element of $(\Lambda^+, <)$ and hence cannot be minuscule (Proposition 1.12).

Conversely, if ω_i is not minuscule then it cannot be minimal (again Proposition 1.12), so by Theorem 2.6 there must be some $\alpha \in E(\Phi)$ such that $\omega_i - \alpha$ is dominant. Since $\alpha > 0$ there must be some index j such that $\langle \alpha, \alpha_j^\vee \rangle > 0$, so this is possible only if $\langle \alpha, \alpha_j^\vee \rangle \leq 0$ for all $j \neq i$ and $\langle \alpha, \alpha_i^\vee \rangle = 1$. Setting $I = \operatorname{Supp} \alpha$, it cannot be the case that Φ_I is a root system of type A, since in that case we would have either $\langle \alpha, \alpha_j^\vee \rangle = 2$ (if $|I| = 1$) or there would be two indices j such that $\langle \alpha, \alpha_j^\vee \rangle = 1$ (the two end nodes of I, if $|I| > 1$). It follows that $\alpha \in E^*(\Phi)$ and $i = p(\alpha)$.

To complete the proof, it remains to be shown that the map is injective. For this we have no found no alternative to using the classification of finite root systems. In the case of G_2, one notes that if α is exceptional, then $p(\alpha)$ indexes the long simple root and $p(\bar{\alpha})$ the short simple root. Otherwise, using the fact that $p(\bar{\alpha}_I)$ is the (unique) node adjacent to the "extra" node in the extended diagram of $\Phi_I^\vee$, this can be established by a simple graph-theoretic analysis of the extended diagrams of the irreducible root systems (see the Appendix of [**B1**]). We leave the details to the reader. □

Corollary 2.11. *Assume Φ is irreducible.*

(a) *We have $|E^*(\Phi)| = n - f + 1$.*
(b) *If Φ is simply-laced, then $|E(\Phi)| = \binom{n+2}{2} - f$.*
(c) *If the diagram of Φ is linear, then $|E(\Phi)| = \binom{n+1}{2}$ ($+1$ if $\Phi = G_2$).*

Proof. (a) If Φ is irreducible, then every minuscule weight is a fundamental weight. This follows from the fact that $\langle \omega_i, \bar{\alpha}^\vee \rangle \geq 1$ for all i ($\bar{\alpha}$ has full support), whence $\langle \lambda, \bar{\alpha}^\vee \rangle \geq 2$ if $\lambda \in \Lambda^+$ is not a fundamental weight. Also, the number of minuscule weights is $f - 1$ (Corollary 1.13), so the number of non-minuscule fundamental weights is $n - f + 1$.

(b) The members of $E(\Phi) - E^*(\Phi)$ are in one-to-one correspondence with the irreducible parabolic subsystems of Φ of type A. If Φ is simply-laced, this is the number of paths in the Dynkin diagram of Φ. However there are $\binom{n+1}{2}$ paths in any tree with n nodes, so the cardinality of $E(\Phi)$ is $\binom{n+1}{2} + (n - f + 1)$.

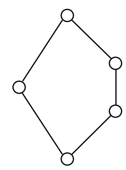

Figure 1.

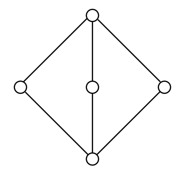

Figure 2.

(c) The locally short dominant roots are in one-to-one correspondence with the connected subgraphs of the Dynkin diagram of Φ. If this diagram is linear, the number of such subgraphs is clearly $\binom{n+1}{2}$. $\square$

3. Grading, Distributivity, and the Lack Thereof

Fix $I \subseteq \{1, \ldots, n\}$, and let Λ_I denote the weight lattice of Φ_I. There is a natural map $\Lambda \to \Lambda_I$, denoted $\lambda \mapsto \lambda'$, that can be defined by the property that $\langle \lambda, \alpha^\vee \rangle = \langle \lambda', \alpha^\vee \rangle$ for all $\alpha \in \Phi_I$. In particular, $\omega_i' = 0$ for $i \notin I$, and $\{\omega_i' : i \in I\}$ is the set of fundamental weights of Φ_I.

Lemma 3.1. *If $\lambda, \mu \in \Lambda^+$ and $\lambda - \mu \in \mathbf{N}\Phi_I^+$, then the subinterval $[\mu, \lambda]$ of $(\Lambda^+, <)$ is isomorphic to the subinterval $[\mu', \lambda']$ of $(\Lambda_I^+, <)$.*

Proof. Since $\beta = \beta'$ for all $\beta \in \mathbf{Z}\Phi_I$, the map $\nu \mapsto \nu'$ is an isomorphism between the subinterval $[\mu, \lambda]$ of $(\Lambda, <)$ and $[\mu', \lambda']$ of $(\Lambda_I, <)$. It therefore suffices to show that if $\mu \leq \nu \leq \lambda$, then ν is dominant if and only if ν' is dominant. Indeed if $\mu \leq \nu \leq \lambda$, then we have $\nu = \lambda - \beta$ for some $\beta \in \mathbf{N}\Phi^+$ with $\operatorname{Supp}\beta \subseteq I$, so the result follows from Lemma 2.4. $\square$

The following result shows that if Φ is irreducible and of rank at least 3, then the lattices $(\Lambda_i^+, <)$ are not graded, and hence not semimodular, or modular, or distributive.

Theorem 3.2. *If Φ is irreducible and of rank $n \geq 3$, then each component of $(\Lambda^+, <)$ has infinitely many subintervals isomorphic to the lattice in Figure 1.*

Proof. Choose a coset Λ_i of Λ and a subset I of $\{1, \ldots, n\}$ so that Φ_I is irreducible and of rank 3. The image of Λ_i with respect to the map $\lambda \mapsto \lambda'$ is a union of cosets of Λ_I modulo $\mathbf{Z}\Phi_I$. Thus if $[\mu - \beta, \mu]$ is a subinterval of $(\Lambda_I^+, <)$ that belongs to one of these cosets, then we can choose a preimage λ of μ in Λ_i^+. Any such preimage will have $\lambda - \beta$ dominant (Lemma 2.4), and furthermore, the subinterval $[\lambda - \beta, \lambda]$ of $(\Lambda_i^+, <)$ will be isomorphic to the subinterval $[\mu - \beta, \mu]$ (Lemma 3.1). Thus it suffices to restrict our attention to $\Phi = A_3$, B_3, and C_3—the irreducible root systems of rank 3.

Arrange the simple roots $\alpha_1, \alpha_2, \alpha_3$ in a linear order consistent with the diagram of Φ, with α_1 short, so that $\langle \alpha_i, \alpha_j^\vee \rangle = 0$ if $|i-j| > 1$ and $\langle \alpha_i, \alpha_j^\vee \rangle = -1$ if $|i-j| = 1$, except that $\langle \alpha_2, \alpha_1^\vee \rangle = -2$ in B_3 and $\langle \alpha_3, \alpha_2^\vee \rangle = -2$ in C_3. Let $\lambda = (m+2)\omega_1 + \omega_2 + \omega_3 \in \Lambda^+$ for some integer $m \geq 0$. We claim that $\mu = \lambda - \alpha_1 - \alpha_2 - \alpha_3$ is dominant, and that the subinterval $[\mu, \lambda]$ of $(\Lambda^+, <)$ is isomorphic to the lattice in Figure 1. The weight coordinates

	A_3	B_3	C_3
$\lambda - \alpha_1 =$	$m\omega_1 + 2\omega_2 + \omega_3$	$m\omega_1 + 2\omega_2 + \omega_3$	$m\omega_1 + 2\omega_2 + \omega_3$
$\lambda - \alpha_1 - \alpha_2 =$	$(m+1)\omega_1 + 2\omega_3$	$(m+2)\omega_1 + 2\omega_3$	$(m+1)\omega_1 + 2\omega_3$
$\lambda - \alpha_2 - \alpha_3 =$	$(m+3)\omega_1$	$(m+4)\omega_1$	$(m+3)\omega_1 + \omega_2$

show that $\lambda - \alpha_1, \lambda - \alpha_1 - \alpha_2, \lambda - \alpha_2 - \alpha_3 \in \Lambda^+$, and therefore $\mu = (\lambda - \alpha_1) \wedge (\lambda - \alpha_2 - \alpha_3)$ is also dominant (Theorem 1.3). The only other elements in the subinterval $[\mu, \lambda]$ of $(\Lambda, <)$ are $\lambda - \alpha_2$, $\lambda - \alpha_3$, and $\lambda - \alpha_1 - \alpha_3$. However these weights are not dominant, since $\langle \lambda - \alpha_2, \alpha_2^\vee \rangle = -1$ and $\langle \lambda - \alpha_1 - \alpha_3, \alpha_3^\vee \rangle = \langle \lambda - \alpha_3, \alpha_3^\vee \rangle = -1$. Hence the subinterval $[\mu, \lambda]$ of $(\Lambda^+, <)$ consists of the five elements $\{\lambda, \mu, \lambda - \alpha_1, \lambda - \alpha_1 - \alpha_2, \lambda - \alpha_2 - \alpha_3\}$, and it is clear that the subposet they form is isomorphic to Figure 1.

Lastly, note that in each case $\Lambda / \mathbf{Z}\Phi$ is a cyclic group generated by ω_1. Therefore as m varies over integers ≥ 0, the subinterval $[\mu, \lambda]$ occurs in each coset of Λ infinitely often according to the congruence class of m mod 4 (in A_3) or mod 2 (otherwise). ☐

THEOREM 3.3. *If Φ is of rank $n \leq 2$, then each component of $(\Lambda^+, <)$ is a sublattice of the corresponding component of $(\Lambda, <)$, and hence distributive (and graded).*

Proof. Given Theorem 1.3, it is necessary and sufficient to show that $\mu \vee \nu$ is dominant for all dominant μ, ν in the same coset of Λ, where $\vee$ denotes the join operation defined by (1.2). By a dual form of Lemma 1.2, we have that for $\gamma \in \mathbf{Z}\Phi^\vee$,

$$\langle \mu, \gamma \rangle, \langle \nu, \gamma \rangle \geq 0 \quad \Rightarrow \quad \langle \mu \vee \nu, \gamma \rangle \geq 0$$

for all μ, ν in the same coset of Λ if and only if there is at most one index i such that $\langle \alpha_i, \gamma \rangle < 0$. Taking $\gamma = \alpha_j^\vee$, we see that the desired conclusion follows if there is at most one negative entry in each column of the Cartan matrix (cf. Remark 1.5(a)). This is clearly true if (and if Φ is irreducible, only if) Φ is of rank at most 2. ☐

Let μ be a minuscule weight or zero; i.e., a minimal element of $(\Lambda^+, <)$. It will be convenient for what follows to introduce the notation $\Phi(\mu)$ for the lattice formed by the component of $(\Lambda^+, <)$ with minimum element μ. If μ is a fundamental weight ω_i, we may also use the abbreviation $\Phi(i)$.

Remark 3.4. (a) Any symmetry of the diagram of Φ induces an automorphism of the semigroup Λ^+, and hence an automorphism of $(\Lambda^+, <)$. In particular, the automorphism permutes the components of $(\Lambda^+, <)$, and hence provides an automorphism of $\Phi(\mu)$ if and only if the automorphism fixes μ. For example, in the case $\Phi = D_4$, the lattice $D_4(0)$ has S_3-symmetry, and the remaining three components of $(\Lambda^+, <)$ are mutually isomorphic.

(b) Not all isomorphisms among the lattices $\Phi(\mu)$ arise from diagram symmetries. For example, if $\Phi = B_n$ and α_1 is short, then ω_1 is the unique minuscule weight of Φ, and we claim that translation by ω_1 is an isomorphism $B_n(0) \to B_n(1)$. Since translation by any dominant weight is clearly an order-preserving map, this amounts to the assertion that $\lambda \in \mathbf{Z}\Phi$ is dominant if and only if $\lambda + \omega_1$ is dominant. However this in turn follows from the reasoning in Case III of Theorem 2.6 (see (2.2)): since $\langle \gamma, \alpha_1^\vee \rangle$ is *even* for all $\gamma \in \mathbf{Z}\Phi$, $\langle \gamma, \alpha_1^\vee \rangle \geq -1$ implies $\langle \gamma, \alpha_1^\vee \rangle \geq 0$.

By the fundamental theorem on distributive lattices (e.g., [**S**]), one knows that a distributive lattice is isomorphic to the lattice of order ideals of the subposet formed by the join-irreducible elements. Consequently, it is of interest to determine these posets of join-irreducibles for the distributive lattices identified by Theorem 3.3. Setting aside the rank one case as trivial (the two components of $(\Lambda^+, <)$ are total orders), let us consider the irreducible root systems of rank 2.

Ordering the simple roots so that α_1 is short, Remark 3.4 shows that there are only four lattices to consider: $A_2(0)$, $A_2(1) \cong A_2(2)$, $B_2(0) \cong B_2(1)$, and $G_2(0)$. Furthermore, in each case the set of generators of the partial order (as in Section 2) is given by

$$E(\Phi) = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2, \bar{\alpha}\},$$

although $\bar{\alpha} = \alpha_1 + \alpha_2$ in case $\Phi = A_2$ or B_2. Since $\alpha_1, \alpha_2 < \alpha_1 + \alpha_2 \leq \bar{\alpha}$, it follows from Theorem 2.6 that $\lambda = m_1\omega_1 + m_2\omega_2 \in \Lambda^+$ is join-irreducible (or a minimal element) if and only if $\lambda - \alpha_1$ and $\lambda - \alpha_2$ are not both dominant; i.e., $\min(m_1, m_2) \leq 1$. Partitioning these weights into the appropriate cosets and deleting the minimal element from each, we obtain the following sets of join-irreducible elements:

$$\begin{aligned}
A_2(0):&\quad \{3m\omega_1, 3m\omega_2, (3m-2)\omega_1 + \omega_2, \omega_1 + (3m-2)\omega_2 : m \geq 1\},\\
A_2(1):&\quad \{(3m+1)\omega_1, (3m-1)\omega_2, (3m-1)\omega_1 + \omega_2, \omega_1 + 3m\omega_2 : m \geq 1\},\\
B_2(0):&\quad \{m\omega_2, 2m\omega_1, 2m\omega_1 + \omega_2 : m \geq 1\},\\
G_2(0):&\quad \{m\omega_1, m\omega_2, m\omega_1 + \omega_2, \omega_1 + m\omega_2 : m \geq 1\}.
\end{aligned}$$

Finite portions of each of the corresponding subposets of $(\Lambda^+, <)$ are displayed in Figure 3.

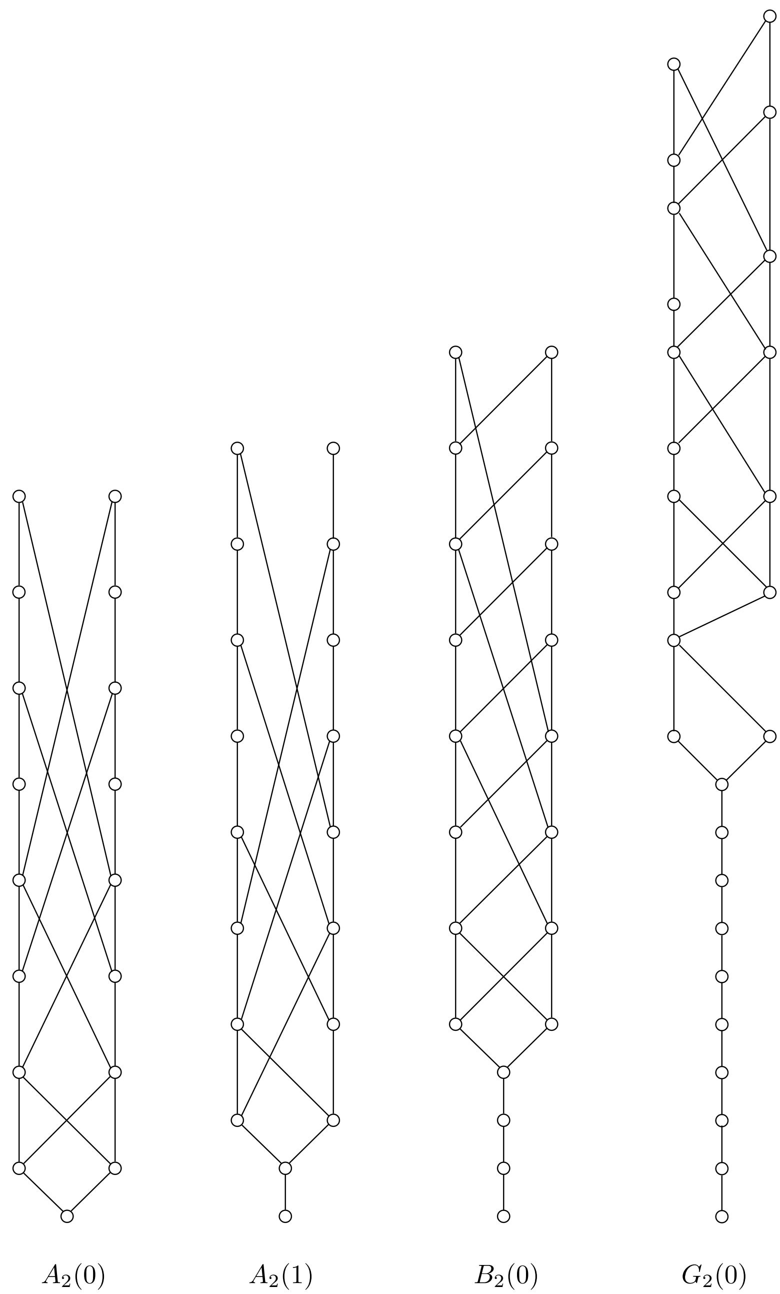

FIGURE 3: Join-irreducibles in rank two.

In each case, the poset of join-irreducibles can be described as a union of two (not necessarily disjoint) chains $a_0 < a_1 < a_2 < \cdots$ and $b_0 < b_1 < b_2 < \cdots$, together with the transitive consequences of the relations

$$\begin{aligned} A_2(0): &\quad b_i \le a_{2i},\ a_i \le b_{2i}, \\ A_2(1): &\quad b_i \le a_{2i},\ a_i \le b_{2i+1}, \\ B_2(0): &\quad b_i \le a_{2i},\ a_i \le b_i, \\ G_2(0): &\quad b_{i+1} \le a_{2i},\ a_{3i} \le b_{2i+1},\ a_{3i+2} \le b_{2i+2} \end{aligned}$$

for all $i \ge 0$. Among these consequences are the equalities $a_0 = b_0$ in $A_2(0)$ and $B_2(0)$, and $a_0 = b_1$, $a_2 = b_2$ in $G_2(0)$.

4. The Möbius Function

Recall that for root systems of type A, the partial order $(\Lambda^+, <)$ is closely related to the dominance order on partitions. By a theorem of Brylawski [**Br**] (see also [**G**]), the latter is known to be totally unimodular, meaning that the Möbius function takes on only the values $\{0, \pm 1\}$. (For an introduction to Möbius functions, see Chapter 3 of [**S**].) In fact, not only is it true that the dominance order on partitions of n is a subinterval of $(\Lambda^+, <)$ for $\Phi = A_{n-1}$, but conversely, every subinterval of $(\Lambda^+, <)$ in type A is isomorphic to a subinterval of the dominance order of partitions of m for some m. Hence $(\Lambda^+, <)$ is also totally unimodular in type A, and this fact is equivalent to Brylawski's result.

Theorem 4.1. *If Φ is irreducible, then the values of the Möbius function of $(\Lambda^+, <)$ are restricted to $\{0, \pm 1, \pm 2\}$. Furthermore, the values ± 2 occur only if $\Phi = D_n$ or E_n.*

We will obtain the above theorem as a corollary of the more general Theorem 4.6 below.

Remark 4.2. (a) Set $\Phi = D_4$ and let γ denote the sum of the four simple roots. It is not hard to show that $\bar{\alpha} + \gamma$ is dominant, and that the subinterval $[\bar{\alpha}, \bar{\alpha} + \gamma]$ of $D_4(0)$ consists of five elements: $\bar{\alpha}$, $\bar{\alpha} + \gamma$, and $\bar{\alpha} + \alpha_i$, where i ranges over the indices of the three end nodes. Hence this subinterval is isomorphic to the lattice in Figure 2 and has Möbius function 2. By the reasoning in Section 3 (see especially Lemma 3.1 and the proof of Theorem 3.2), it follows that for any root system Φ that *properly* contains D_4 as a parabolic subsystem Φ_I, there are infinitely many subintervals with Möbius function 2 in the components of $(\Lambda^+, <)$ whose image under the map $\Lambda \to \Lambda_I$ contains the trivial coset $\mathbf{Z}\Phi_I$. This includes every component in the cases of D_n ($n \ge 5$) and E_n except for the components of the two minuscule weights at the forked end of D_n.

(b) For intervals with Möbius function -2, consider $\Phi = D_5$ with the nodes numbered in the form $\substack{1\\2}345$. For any integer $m \ge 1$, the subinterval of $(\Lambda^+, <)$ from $\mu = \omega_3 + \omega_4 + m\omega_5$

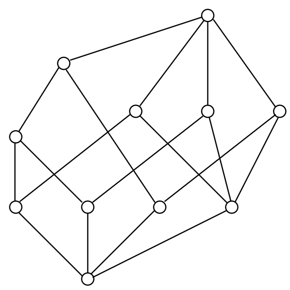

FIGURE 4.

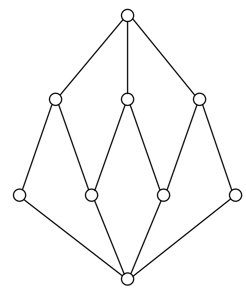

FIGURE 5.

to $\lambda = \omega_1 + \omega_2 + \omega_4 + (m+1)\omega_5$ is isomorphic to the lattice in Figure 4, and hence has Möbius function -2. This subinterval belongs to $D_5(0)$ or $D_5(5)$ according to the parity of m, so the reasoning in (a) shows that all components in the case $\Phi = E_n$, and two of the components in the case $\Phi = D_n$ $(n \geq 5)$, have infinitely many subintervals with Möbius function -2.

4.1 The Möbius Algebra.

Let L be a finite join-semilattice (including $\hat{0}$), and let $\mathbf{Z}[L]$ denote the semigroup ring of L. Thus $\mathbf{Z}[L]$ is freely generated as an abelian group by the members of L, and the multiplication is such that $(x, y) \mapsto x \vee y$ for $x, y \in L$. Note that $\hat{0} = 1$; i.e., the minimum element of L is a unit element for the ring $\mathbf{Z}[L]$.

Define elements $e_x \in \mathbf{Z}[L]$ for each $x \in L$ so that

$$e_x = \sum_{y \geq x} \mu(x, y) y,$$

where μ denotes the Möbius function of L. By Möbius inversion, we have

$$x = \sum_{y \geq x} e_y.$$

The following result is due to Solomon [**So**] (see also Theorem 3.9.2 of [**S**]).

PROPOSITION 4.3. *The elements e_x are orthogonal idempotents (i.e., $e_x \vee e_y = \delta_{xy} e_x$), and thus $\mathbf{Z}[L]$ is ring-isomorphic to a direct sum of $|L|$ copies of $\mathbf{Z}$.*

Proof. Define a (possibly) new product on $\mathbf{Z}[L]$ by setting $e_x * e_y = \delta_{xy} e_x$ for $x, y \in L$. For this product, we have

$$x * y = \sum_{z \geq x} e_z * \sum_{z \geq y} e_z = \sum_{z \geq x, y} e_z = x \vee y,$$

so this is in fact the defining product for $\mathbf{Z}[L]$. $\square$

The following is a version of Weisner's Theorem (cf. Corollary 3.9.4 of [**S**]).

PROPOSITION 4.4. *If $\{a_1, \ldots, a_l\} \subseteq L - \{\hat{0}\}$ includes the atoms of L, then*

$$(1 - a_1) \vee \cdots \vee (1 - a_l) = \sum_{x \in L} \mu(\hat{0}, x) x = e_{\hat{0}}.$$

Proof. We have $1 - a = \hat{0} - a = \sum_{x \not\geq a} e_x$. By Proposition 4.3, it follows that for *any* $a_1, \ldots, a_l$, the coefficient of e_x in $(1 - a_1) \vee \cdots \vee (1 - a_l)$ is 0 or 1, the latter occurring if and only if $x \not\geq a_1, \ldots, x \not\geq a_l$. If every atom occurs among the a_i's, then $x = \hat{0}$ is the only member of L with this property. □

4.2 Semilattices in $\mathbf{N}\Phi^+$.

Given any finite subset $\mathcal{B} \subset \mathbf{N}\Phi^+ - \{0\}$, let $L(\mathcal{B})$ denote the join-semilattice generated by $\mathcal{B}$. Thus $L(\mathcal{B})$ consists of the subposet of $(\mathbf{N}\Phi^+, <)$ formed by the joins of all subsets of $\mathcal{B}$. The posets $L(\mathcal{B})$ are equivalent to the "lattices of multisets" studied by Greene in [**G**, §4]: Each $\beta = \sum b_i \alpha_i \in \mathbf{N}\Phi^+$ corresponds to a multiset in which i occurs with multiplicity b_i. In this correspondence, joins in $(\mathbf{N}\Phi^+, <)$ correspond to multiset unions.

Let $\mu_{\mathcal{B}}$ denote the value of the Möbius function of $L(\mathcal{B})$ from $\hat{0}$ to $\hat{1}$. Let us also define $\beta|_i := b_i$ if $\beta = \sum_i b_i \alpha_i \in \mathbf{Z}\Phi$.

We define $\mathcal{B}$ to be *reducible* if either of the following holds:

I. $\beta < \beta'$ for some $\beta, \beta' \in \mathcal{B}$.

In this case, working in the Möbius algebra $\mathbf{Z}[L(\mathcal{B})]$, we have

$$(1 - \beta) \vee (1 - \beta') = 1 - \beta - \beta' + \beta \vee \beta' = 1 - \beta.$$

Setting $\mathcal{B}' = \mathcal{B} - \{\beta'\}$, it follows from Proposition 4.4 that either $\mu_{\mathcal{B}} = \mu_{\mathcal{B}'}$ or $\mu_{\mathcal{B}} = 0$, according to whether $L(\mathcal{B}')$ includes the maximum element of $L(\mathcal{B})$.

II. There exists $\beta \in \mathcal{B}$ and an index i such that $\beta|_i > \beta'|_i$ for all $\beta' \in \mathcal{B} - \{\beta\}$.

More explicitly, suppose that $\mathcal{B} = \{\beta_1, \ldots, \beta_l\}$, $\beta = \beta_1$, and that $\bar{\beta} = \beta_1 \vee \cdots \vee \beta_l$ is the maximum element of $L(\mathcal{B})$. Given the hypotheses, we have $\beta_{i_1} \vee \cdots \vee \beta_{i_k} = \bar{\beta}$ if and only if 1 occurs among the indices $i_1, \ldots, i_k$ (say $i_1 = 1$), and $\beta_{i_2}|_I \vee \cdots \vee \beta_{i_k}|_I = \bar{\beta}|_I$, where $I = \{j : \beta_1|_j < \bar{\beta}|_j\}$. It follows from Proposition 4.4 that $\mu_{\mathcal{B}} = -\mu_{\mathcal{B}'}$, where $\mathcal{B}' = \{\beta_2|_I, \ldots, \beta_l|_I\}$. Note that we may insist that the members of $\mathcal{B}'$ are nonzero, since $\mathcal{B}$ is otherwise reducible in the sense of I.

In either case, we refer to $\mathcal{B}'$ as a *simple reduction* of $\mathcal{B}$. More generally, if $\mathcal{B}'$ can be obtained from $\mathcal{B}$ by a sequence of zero or more simple reductions, then we say that $\mathcal{B}'$ is a *reduction* of $\mathcal{B}$. In such cases, the above analysis shows that $\mu_{\mathcal{B}} = \pm\mu_{\mathcal{B}'}$ or $\mu_{\mathcal{B}} = 0$.

4.3 Elementary Semilattices.

To explain the relevance of the semilattices $L(\mathcal{B})$ for computing the Möbius function of $(\Lambda^+, <)$, recall from Theorem 2.6 that $E(\Phi) \subseteq \Phi^+$ is the set of generators of $(\Lambda^+, <)$, in the sense that if λ covers μ, then $\lambda - \mu \in E(\Phi)$.

The following can be viewed as a generalization of Lemma 3.1 of [**G**] to root systems.

LEMMA 4.5. *Every subinterval of $(\Lambda^+, <)$ is dually isomorphic to $L(\mathcal{B})$ for some subset $\mathcal{B}$ of $\mathbf{N}\Phi^+ - \{0\}$. Furthermore, if $\mu_{\mathcal{B}} \neq 0$, then $\mu_{\mathcal{B}} = \mu_{\mathcal{B}'}$ for some $\mathcal{B}' \subseteq E(\Phi)$.*

Proof. Consider an arbitrary subinterval $[\mu, \lambda]$ of $(\Lambda^+, <)$. The map $\nu \mapsto \lambda - \nu$ defines a dual embedding of $[\mu, \lambda]$ as a subposet of $(\mathbf{N}\Phi^+, <)$. By Theorem 1.3, this map carries the meet operation of $[\mu, \lambda]$ to the join operation of $(\mathbf{N}\Phi^+, <)$, so $[\mu, \lambda] \cong L(\mathcal{B})^*$, where $\mathcal{B} = \{\lambda - \nu : \mu \leq \nu < \lambda,\ \nu \in \Lambda^+\}$. Now by Theorem 2.6, the set $\mathcal{B}'$ of atoms of $L(\mathcal{B})$ is a subset of $E(\Phi)$, and since $\mathcal{B}'$ can be obtained from $\mathcal{B}$ by a sequence of simple reductions of type I, we have either $\mu_{\mathcal{B}} = 0$ or $\mu_{\mathcal{B}} = \mu_{\mathcal{B}'}$. □

To prove Theorem 4.1, the previous lemma shows that it is sufficient to determine $\mu_{\mathcal{B}}$ for all $\mathcal{B} \subseteq E(\Phi)$. It should be noted however that not all such subsets, even those whose members are pairwise incomparable, are realizable in the sense that there is a subinterval $[\mu, \lambda]$ of $(\Lambda^+, <)$ whose co-atom set is $\{\lambda - \alpha : \alpha \in \mathcal{B}\}$.

The following result can be viewed as a generalization of Theorem 2.1 of [**G**] from root systems of type A to general root systems.

THEOREM 4.6. *If Φ is irreducible, then for every $\mathcal{B} \subseteq E(\Phi)$, we have $\mu_{\mathcal{B}} \in \{0, \pm 1, \pm 2\}$. Furthermore, if $\mu_{\mathcal{B}} = \pm 2$, then there is a reduction $\mathcal{B}' = \{\bar{\alpha}_I, \bar{\alpha}_J, \bar{\alpha}_K\}$ of $\mathcal{B}$ in which each of Φ_I, Φ_J, Φ_K are of type A and $L(\mathcal{B}')$ is isomorphic to the lattice in Figure 2.*

Proof. Proceed by induction on $|\mathcal{B}| + \operatorname{rank} \Phi$, the base of the induction being the trivial case in which $\mathcal{B}$ is empty. We may assume that for each end node i of the diagram of Φ there exists $\beta \in \mathcal{B}$ with $i \in \operatorname{Supp} \beta$. If not, we can replace Φ with an irreducible subsystem of lower rank. We may also assume that the members of $\mathcal{B}$ are pairwise incomparable, since otherwise $\mu_{\mathcal{B}} = 0$ or a simple reduction of type I may be applied, deleting a member of $\mathcal{B}$ and at the same time preserving the value of the Möbius function. On the other hand, we cannot immediately eliminate the possibility that $\mathcal{B}$ has a reduction of type II, since a reduction of this type might fail to yield a subset of $E(\Phi)$.

Declare $\beta \in \mathbf{N}\Phi^+$ to be *thin* if $\beta|_i \leq 1$ for all i; otherwise β is *fat*. Note that a locally short dominant root $\bar{\alpha}_I$ is thin if and only if Φ_I is a root system of type A or B.

Case I: The diagram of Φ is a path. In this case, the fat roots in $E(\Phi)$ are the locally short dominant roots corresponding to parabolic subsystems of type C, F_4, and G_2. In particular, if $\Phi = C_n$ there is a fat root corresponding to each of the subsystems $C_3, \dots, C_n$; in F_4 there is one each corresponding to C_3 and F_4, and in G_2 there is only the short dominant root itself. In each case, the "fat" parabolic subsystems are totally ordered by inclusion, so the fat roots in $E(\Phi)$ are totally ordered with respect to $<$. Since the members of $\mathcal{B}$ are pairwise incomparable, there can be at most one fat root in $\mathcal{B}$.

Now let i be an end node of Φ and α a member of $\mathcal{B}$ whose support includes i. The above case analysis shows that each fat root in $E(\Phi)$ has support that contains at least one end node, so we can insist that i and α are chosen so that all members of $\mathcal{B}$, except possibly α, are thin.

Since the diagram of Φ is a path, the set of irreducible parabolic subsystems of Φ that include α_i are totally ordered by inclusion, so the corresponding locally short dominant roots are totally ordered with respect to $<$. (In G_2 there is an exceptional root, but it is still the case that the members of $E(\Phi)$ with support including a fixed end node are totally ordered.) Hence α is the *unique* member of $\mathcal{B}$ whose support includes i. Furthermore, since the remaining members of $\mathcal{B}$ are thin, we have $\alpha|_j \geq \beta|_j$ for all $j \in \operatorname{Supp} \alpha$ and $\beta \in \mathcal{B}$. It follows that we can apply a simple reduction of type II, deleting α from $\mathcal{B}$ and restricting each of the remaining members of $\mathcal{B}$ to $I = \operatorname{Supp}(\alpha)^c$. However I spans a connected subgraph of the diagram of Φ, so the restriction $\beta|_I$ of a thin $\beta \in \mathcal{B} - \{\alpha\}$ is a (thin) member of $E(\Phi)$. It follows by induction that $\mu_{\mathcal{B}} \in \{0, \pm 1\}$.

Case II: The diagram of Φ has a fork (i.e., $\Phi = D_n$ or E_n). In this case, let us allow 0 as the index of a simple root and view the diagram of Φ as a subgraph of

$$\begin{array}{c} \quad\;\; 1 \\ \quad\;\; | \\ 0 - 2 - 3 - 4 - 5 - 6 - 7 \;\cdots . \end{array}$$

Thus if $\Phi = D_n$, then the simple roots are indexed by $1, \dots, n$, whereas if $\Phi = E_n$, the indices are $0, 1, \dots, n-1$. For convenience, we will use E_5 as the name of the parabolic subsystem generated by the simple roots indexed by $0, 1, \dots, 4$, even though it is isomorphic to D_5. With this convention, the parabolic subsystems of type D (respectively, type E) are totally ordered by inclusion, so there can be at most one locally dominant root of type D and one of type E in $\mathcal{B}$.

First consider the possibility that $\mathcal{B}$ includes the locally dominant root α of type E_r. One can check that for all locally dominant roots β of types A and D, we have $\alpha|_2 \geq 2 > \beta|_2$. Moreover, for all $i \in \operatorname{Supp} \alpha$ we have $\alpha|_i \geq \beta|_i$ except possibly when β is of type D and $i = r - 1$. It follows that $\mathcal{B}$ has a simple reduction $\mathcal{B}' = \{\beta|_I : \beta \in \mathcal{B} - \{\alpha\}\}$ where $I = \{r, \dots, n-1\}$ or $\{r-1, \dots, n-1\}$. However if $\mathcal{B}$ includes a locally dominant root β of type D, then $\beta|_I$ may fail to be a member of $E(\Phi)$. In that case $\beta|_I$ is fat, has support that includes an end node of Φ_I, and all other members of $\mathcal{B}'$ are thin roots in $E(\Phi)$. Hence, a second reduction of type II can be applied, yielding a configuration of thin roots in a subsystem of Φ_I, which is of type A. Otherwise, $\mathcal{B}'$ is already of this form, so in either case we obtain $\mu_{\mathcal{B}} \in \{0, \pm 1\}$, by the reasoning of Case I.

Next suppose that $\mathcal{B}$ includes the locally dominant root α of type D_r, but no locally dominant root of type E. Since all remaining members of $\mathcal{B}$ must be thin, we have $\alpha|_i \geq \beta|_i$ for all $i \in \operatorname{Supp}\alpha$, so $\mathcal{B}' = \{\beta|_I : \beta \in \mathcal{B} - \{\alpha\}\}$ is a reduction of $\mathcal{B}$, where $I = \operatorname{Supp}(\alpha)^c$. If $\Phi = D_n$, then $\mathcal{B}'$ is a set of thin roots in Φ_I (an irreducible subsystem of type A), so as in the previous case, we conclude that $\mu_{\mathcal{B}} \in \{0, \pm 1\}$. On the other hand, if $\Phi = E_n$ then $I = \{0, r+1, \dots, n-1\}$, Φ_I is not necessarily irreducible, and the members of $\mathcal{B}'$ need not be roots. However in that case, the permutation of the simple roots that interchanges α_0 and α_r induces a permutation of $\mathbf{N}\Phi^+$ that preserves the isomorphism class of $L(\mathcal{B}')$, but at the same time maps $\mathcal{B}'$ to a set of (thin) roots in the type A subsystem indexed by $\{r, r+1, \dots, n-1\}$. So again by induction, we obtain $\mu_{\mathcal{B}} \in \{0, \pm 1\}$.

Henceforth we may assume that all members of $\mathcal{B}$ are (thin) locally dominant roots of type A. If there is an end node that occurs in the support of only one root $\alpha \in \mathcal{B}$, then we can apply a reduction of type II in which α is deleted from $\mathcal{B}$ and the remaining members are restricted to $I = \operatorname{Supp}(\alpha)^c$. If $3 \notin \operatorname{Supp}\alpha$, then I spans a connected subgraph of the diagram of Φ, so the members of the reduction $\mathcal{B}'$ are again thin locally dominant roots and the induction continues. However if $3 \in \operatorname{Supp}\alpha$, then I may have two connected components and the members of $\mathcal{B}'$ need not be roots. In that case, there is a permutation of the simple roots that merges the two components into a single path and maps $\mathcal{B}'$ to a set of roots in this root subsystem of type A. Thus we again obtain $\mu_{\mathcal{B}} \in \{0, \pm 1\}$.

The remaining possibility is that every end node appears in the support of at least two members of $\mathcal{B}$. Since the support of a thin root is a path in the diagram of Φ, the fact that the members of $\mathcal{B}$ are pairwise incomparable implies that for each end node i, there are *exactly* two members of $\mathcal{B}$ whose support paths include i, and these paths must end in distinct branches of the diagram. However, (at least) one of the branches has only one node, so two of the supporting paths must be $I = \{2, 3, 1\}$ and $J = \{1, 3, 4, \dots, n\}$ (if $\Phi = D_n$), or $I = \{0, 2, 3, 1\}$ and $J = \{1, 3, 4, \dots, n-1\}$ (if $\Phi = E_n$). For the remainder of $\mathcal{B}$ there are only two possibilities: (1) there is one additional member, a thin root whose support K is the remaining path between end nodes of the diagram of Φ, or (2) $\Phi = E_n$ and there are two additional members, consisting of thin roots whose support paths are $K = \{0, 2, 3, \dots, r\}$ and $L = \{2, 3, 4, \dots, n-1\}$, where $4 \leq r \leq n-2$. In the former case, $L(\mathcal{B})$ is isomorphic to the lattice in Figure 2, and hence has Möbius function 2. In the latter case, $L(\mathcal{B})$ is isomorphic to the lattice in Figure 5, which has Möbius function 0. $\square$

Remark 4.7. (a) Let $\Phi = D_n$ and let $[\mu, \lambda]$ be a subinterval of $(\Lambda^+, <)$ with Möbius function ± 2. Theorem 4.6 and the proof of Lemma 4.5 show that one of the co-atoms of $[\mu, \lambda]$ must be $\lambda - \alpha$, where $\alpha = \alpha_1 + \alpha_2 + \alpha_3$ is a locally dominant root of type A_3. (The simple roots are indexed as in Case II of the above argument.) Since $\langle \alpha, \alpha_1^\vee \rangle = 1$, it follows that $\langle \lambda, \alpha_1^\vee \rangle \geq 1$. In fact $\langle \lambda, \alpha_1^\vee \rangle = 1$, since $\lambda - \alpha_1$ would otherwise be dominant, contradicting the fact that λ covers $\lambda - \alpha$. Similarly, we must have $\langle \lambda, \alpha_2^\vee \rangle = 1$. However, any weight λ for which $\langle \lambda, \alpha_1^\vee \rangle = \langle \lambda, \alpha_2^\vee \rangle$ belongs either to the root lattice or the coset of the minuscule weight ω_n. Hence, the (isomorphic) lattices $D_n(1)$ and $D_n(2)$ corresponding to the remaining components of $(\Lambda^+, <)$ are totally unimodular.

(b) Specializing to the case $\Phi = D_4$, the presence of three-fold symmetry implies that if the subinterval $[\mu, \lambda]$ has Möbius function ± 2, then the interval must have three co-atoms, corresponding to the three locally dominant roots of type A_3. Furthermore, the above reasoning shows that $\langle \lambda, \alpha_i^\vee \rangle = 1$ for $i = 1, 2, 4$. We must also have $\langle \lambda, \alpha_3^\vee \rangle = 0$, since otherwise $\lambda - \alpha_1 - \alpha_2$ would be dominant. Hence $\lambda = \omega_1 + \omega_2 + \omega_4 = 2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4$, and $\mu = \omega_3 = \alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4$ (the meet of the co-atoms). In other words, $[\mu, \lambda]$ is the subinterval identified in Remark 4.2(a).

Remarks 4.2 and 4.7 show that if Φ is irreducible, then the lattice $\Phi(\mu)$ is totally unimodular if and only if the diagram of Φ is a path, or $\Phi = D_n$ and $\mu \in \{\omega_1, \omega_2\}$, or $\Phi = D_4$ and $\mu = \omega_4$. Moreover, if $\Phi(\mu)$ is not totally unimodular, then the Möbius function achieves both of the values 2 and -2 infinitely often, unless $\Phi = D_4$ and $\mu = 0$, in which case there is a unique subinterval with Möbius function 2, and no subinterval with Möbius function -2.

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