A Finiteness Theorem for W-Graphs

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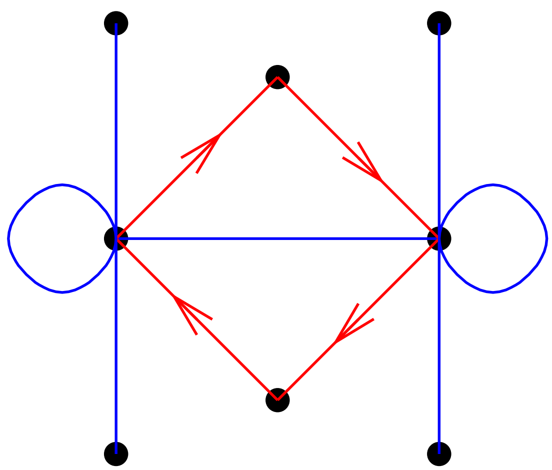
Contents



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1. Introduction

A W-graph for a Coxeter group W is a combinatorial structure that encodes a module for the group algebra of W, or more generally, a module for the associated Iwahori-Hecke algebra. Of special interest are the W-graphs that encode the action of the Hecke algebra on its Kazhdan-Lusztig basis, as well as the action on individual W-cells; i.e. subquotients spanned by the strongly connected components of the W-graph.

As part of a project to understand the essential combinatorial features of the Kazhdan-Lusztig W-graph, we isolated in [**S**] a few of the basic properties of this graph, and used these properties to define the class of "admissible" W-graphs. (Precise definitions are in Section 4 below.) The main idea has been that insight into the structure of admissible W-graphs should lead to insight into the Kazhdan-Lusztig W-graph.

Our main objective in this paper is to prove that for finite W, there are only finitely many admissible W-cells, answering one of the basic questions that we raised earlier in [**S**]. As we shall demonstrate, the crucial feature of admissibility that leads to this conclusion is the fact that admissible W-graphs have nonnegative integer edge weights. A related surprise is that the finiteness is ultimately a consequence of a more fundamental theorem about nonnegative integer matrices.

To outline the contents of the paper, we begin by introducing "polynomial cells" in Section 2. These are matrices with strongly connected support that satisfy a given polynomial identity. The key result (Theorem 2.1) is that for a given polynomial, there are only finitely many such cells over the nonnegative integers. We also give two upper bounds (Theorems 2.3 and 2.5) on the size of a cell. The first of these applies in all cases, whereas the second applies only when the dominant eigenvalue is a rational integer. We also point out that this latter bound is sometimes sharp. For example, there are cubic polynomials for which the bound is attained whenever directed strongly regular graphs with certain parameter sets exist (Proposition 2.8).

In Section 3, we generalize to the case of cells for finite-dimensional associative algebras; the polynomial cells of Section 2 correspond to cells for an algebra with a single generator. As in the polynomial case, we prove that a given algebra has only finitely many cells (up to isomorphism) over the nonnegative integers (Theorem 3.1). In the final section, we apply Theorem 3.1 to the group algebra of a finite Coxeter group W, yielding our main result: a proof that there are only finitely many admissible W-cells, or more generally, W-cells with nonnegative integer edge weights (Theorem 4.1).

A remaining problem of interest is that of constructing all admissible W-cells, or even the simpler problem of finding an effective algorithm for generating polynomial cells.

2. Polynomial cells

Recall that a directed graph is *strongly connected* if there is a (directed) path from any vertex to any other vertex; or equivalently, there is a closed path that passes through every vertex. In particular, a graph with one vertex and no edges is strongly connected.

Given an $n \times n$ matrix $A = [a_{ij}]$, let $G(A)$ denote its support graph; i.e., the directed graph with vertex set $\{1, \dots, n\}$ and edges $i \leftarrow j$ for each pair (i, j) such that $a_{ij} \neq 0$.

Abusing notation, we say that A is strongly connected if $G(A)$ is strongly connected.

Given a nonzero polynomial $p(t) \in \mathbb{Z}[t]$, we define a square matrix A to be a *$p(t)$-cell* if $p(A) = 0$ and A is strongly connected. Note that if $p(A) = 0$ but A is not strongly connected, then there is a simultaneous permutation of the rows and columns of A that has a block triangular form in which each diagonal block is a $p(t)$-cell.

THEOREM 2.1. *For each nonzero polynomial $p(t) \in \mathbb{Z}[t]$, there are only finitely many $p(t)$-cells over the nonnegative integers.*

Proof. First we argue that there is an upper bound on the size n of a nonnegative integer $p(t)$-cell A. Assuming p has degree r (say), the relation $p(A) = 0$ implies that A^r is in the linear span of $\{I, A, \dots, A^{r-1}\}$, and more generally by iteration,

$$A^s \in \operatorname{Span}\{I, A, \dots, A^{r-1}\} \quad (s \geqslant 0).$$

Since A is strongly connected and nonnegative, this implies that there is a directed path of length less than r between any two vertices. Equivalently, the integer matrix

$$B := I + A + \cdots + A^{r-1}$$

is positive. Letting J denote the $n \times n$ matrix of 1's, we have $B \geqslant J$ (entry-wise), and

$$B^m \geqslant J^m = n^{m-1} J. \tag{2.1}$$

Thus all entries of B^m grow exponentially as $m \to \infty$.

On the other hand, the eigenvalues of A must be roots of $p(t)$, so the eigenvalues of B are of the form $1 + \lambda + \cdots + \lambda^{r-1}$, where λ ranges over the (finite) list of roots of $p(t)$. Thus the absolute value of the largest eigenvalue of B is at most

$$\rho := \max \left\{ |1 + \lambda + \cdots + \lambda^{r-1}| : p(\lambda) = 0 \right\}, \tag{2.2}$$

and this bound depends only on $p(t)$. Passing to the Jordan Canonical Form of B, one sees that the entries of B^m may grow asymptotically no faster than $O(m^{k-1} \rho^m)$, where k is the size of the largest Jordan block associated to an eigenvalue with absolute value ρ. Comparing this with (2.1), we obtain that $n \leqslant \rho$; i.e., the size of A is bounded.

To complete the proof, we need only to show that the entries of A are also bounded. For this, recall that there is a directed path in $G(A)$ of length $< r$ between any two vertices. Thus either $a_{ij} = 0$ or there is a closed path of length $\leqslant r$ passing through the edge $i \leftarrow j$, and the product of the matrix entries along this closed path is at least a_{ij}. Since the trace of A^k enumerates weighted sums of closed paths of length k, we obtain

$$a_{ij} \leqslant \max \left\{ \operatorname{tr} A^k : 1 \leqslant k \leqslant r \right\} \leqslant \rho \cdot \max \left\{ |\lambda|^k : p(\lambda) = 0,\, 1 \leqslant k \leqslant r \right\}, \tag{2.3}$$

and thus the entries of A are bounded. $\square$

Remark 2.2. (a) Since the eigenvalues of integer matrices are algebraic integers, there is no loss of generality in requiring the polynomial $p(t)$ to be monic.

(b) By the Perron-Frobenius Theorem [**M**], one knows that a strongly connected nonnegative matrix has a dominant real nonnegative eigenvalue, and all extremal eigenvalues are simple roots of the characteristic polynomial. Thus we may further require that the largest roots of $p(t)$ in absolute value are all simple, and that one of them is real and nonnegative. Letting α denote this dominant real root, it follows that the bound on the size of A obtained in (2.2) is

$$\rho = 1 + \alpha + \cdots + \alpha^{r-1}.$$

Furthermore, aside from the degenerate case $\alpha = 0$ (in which case $p(t) = t$ and the only $p(t)$-cell is a 1×1 zero matrix), the dominant root of a monic, integer polynomial cannot be inside the unit disk, so $\alpha \geqslant 1$ and the upper bound in (2.3) simplifies to $a_{ij} \leqslant \rho\alpha^r$.

(c) Another simplifying consequence of the Perron-Frobenius Theorem that we omitted from the above proof is that the Jordan blocks associated to the extremal eigenvalues of B are necessarily one-dimensional.

The following result provides a tighter bound on the size of A.

Theorem 2.3. *If $p(t) = t^r + c_{r-1}t^{r-1} + \cdots + c_s t^s$ is an integer polynomial whose largest real root is $\alpha > 0$, then every nonnegative integer $p(t)$-cell has size at most $\sum_{i \in N} \alpha^i$, where $N = \{i : s \leqslant i < r,\ c_i \leqslant 0\}$.*

Proof. Let A be a nonnegative integer $p(t)$-cell. Excluding the trivial case $A = [\,0\,]$, we claim that the entries of the matrix $B_N := \sum_{i \in N} A^i$ are positive. To see this, note first that the relation $p(A) = 0$ implies that A^r and all higher powers of A are in the linear span of $\{A^s, \ldots, A^{r-1}\}$. Since A is assumed to be strongly connected, it follows that

$$B := A^s + A^{s+1} + \cdots + A^{r-1}$$

is positive. Thus to prove the claim, it suffices to show that the support graph of each matrix A^i such that $c_i > 0$ is a subgraph of the support graph of B_N. For this, rewrite the dependence relation $p(A) = 0$ in the form $p_+(A) = p_-(A)$ so that $p_+(t)$ and $p_-(t)$ have the terms of $p(t)$ with positive and negative coefficients, respectively. The latter terms have degrees that belong to N, so the support graph of $p_-(A)$ is a subgraph of the support graph of B_N. However, the relation $p_+(A) = p_-(A)$ shows that the support graph of each power A^i with $c_i > 0$ is also contained in the support graph of $p_-(A)$, proving the claim.

Now we proceed as in the proof of Theorem 2.1. Since B_N is positive and integral, the entries of B_N^m are at least n^{m-1}, where n denotes the size of A. On the other hand, the dominant eigenvalue of A is at most α, so the dominant eigenvalue of B_N is at most $\beta := \sum_{i \in N} \alpha^i$. The controlling factor in the asymptotic growth rate of the entries of B_N^m is therefore at most $O(\beta^m)$, so we must have $n \leqslant \beta$. $\square$

Example 2.4. The largest real root of

$$p(t) = (t-3)(t^3 - t) = t^4 - 3t^3 - t^2 + 3t$$

occurs at $t = 3$, so Theorem 2.3 implies that the size of the largest nonnegative integer $p(t)$-cell is at most $3^2 + 3^3 = 36$. This is a slight improvement over the bound one obtains from Remark 2.2(b); see Example 2.7 for further improvement.

It is natural to refine the classification of nonnegative integer $p(t)$-cells according to their dominant real eigenvalue. If this eigenvalue is to be α (necessarily a root of $p(t)$), then by the Perron-Frobenius Theorem, one should discard from $p(t)$ all irreducible factors over $\mathbb{Q}[t]$ that involve roots that are greater than α in absolute value, as well as any duplicate factors involving roots that equal α in absolute value.

In this direction, the following result provides an upper bound that is in some cases better than the one in Theorem 2.3, and in some cases it is sharp. However, it requires α to be a rational integer. We do not know if there is a similar bound when α is irrational.

Theorem 2.5. *Let $p(t) \in \mathbb{Z}[t]$ be a monic polynomial whose largest real root is α, and assume without loss of generality that α is a simple root. If α is an integer and A is a nonnegative integer $p(t)$-cell whose dominant eigenvalue is α, then there is a pair of positive integer column vectors v, w such that*

$$q(A) = vw^T \quad \text{and} \quad q(\alpha) = w^T v,$$

where $q(t) = p(t)/(t-\alpha)$. Moreover, the size of A is at most $q(\alpha)$, and equality occurs if only if $q(A) = J$ and A is α-regular (i.e., every row and column of A has sum α).

Proof. Let A be an $n \times n$ nonnegative integer $p(t)$-cell with dominant eigenvalue α, and $v_1, \ldots, v_n$ a basis for $\mathbb{C}^n$ such that the matrix of A with respect to this basis is in Jordan Canonical Form. By the Perron-Frobenius Theorem, we know that α is a simple root of the characteristic polynomial of A, so we may arrange the basis so that v_1 is the unique eigenvector of A with eigenvalue α. All other basis vectors are annihilated by operators $(A - \beta I)^k$ for various divisors $(t - \beta)^k$ of $p(t)$ with $\beta \neq \alpha$.

Thus we have $q(A)v_1 = q(\alpha)v_1$ and $q(A)v_i = 0$ for $i > 1$. We know that $q(t)$ has integer coefficients (recall that α is assumed to be an integer) and $q(\alpha) \neq 0$ by hypothesis, hence $q(A)$ is an integer matrix of rank 1 whose range is spanned by v_1. If we normalize v_1 so that its coordinates are relatively prime integers, then there must be a (unique) integer column vector w_1 such that

$$q(A) = v_1 w_1^T. \tag{2.4}$$

Similarly analyzing the right action of A on row vectors, one sees that w_1^T must be a left eigenvector for A with eigenvalue α. Recognizing that $q(\alpha)$ is the only nonzero eigenvalue of $q(A)$, we obtain

$$q(\alpha) = \operatorname{tr} q(A) = w_1^T v_1. \tag{2.5}$$

By the Perron-Frobenius Theorem, we may replace (v_1, w_1) with $(-v_1, -w_1)$ if necessary so that all coordinates of v_1 are positive. If the coordinates of the dominant left eigenvector w_1^T were not all positive, then again by the Perron-Frobenius Theorem, they would have to be all negative, and hence $q(\alpha) < 0$. However,

$$q(\alpha) = \lim_{t \to \alpha} p(t)/(t - \alpha) = p'(\alpha)$$

and $p(t)$ is positive for sufficiently large t (since $p(t)$ is monic), so having $q(\alpha) = p'(\alpha) < 0$ would force the existence of a real root of $p(t)$ larger than α, a contradiction.

For the bound on the size of A, note that the dot product of w_1 and v_1 is a sum of n positive integers, hence (2.5) implies $n \leqslant q(\alpha)$. If equality occurs, then w_1 and v_1 must be vectors of 1's, and (2.4) implies $q(A) = J$. Since w_1^T and v_1 are left and right eigenvectors with eigenvalue α, they force A to be α-regular. Conversely, if $q(A) = J$, then v_1 and w_1 must be vectors of 1's and $w_1^T v_1$ is the size of A. $\square$

Remark 2.6. (a) As a partial converse to Theorem 2.5, suppose $p(t)$, $q(t)$, and α are as above, and A is a nonnegative integer matrix such that $q(A) = vw^T$ and $q(\alpha) = w^T v$ for some pair of positive integer vectors v, w. Since $q(A)$ is positive, some combination of powers of A has full support; thus A is strongly connected. It follows that A has a unique eigenvector with positive coordinates (up to normalization), by the Perron-Frobenius Theorem. In fact, this eigenvector must be v, since it must also be the unique positive eigenvector of the positive matrix $q(A) = vw^T$. Letting β denote the (dominant, necessarily nonnegative integer) eigenvalue of A associated with v, we see that

$$(A - \beta I)q(A) = (A - \beta I)vw^T = 0,$$

so A is a $(t - \beta)q(t)$-cell, but not necessarily a $p(t)$-cell unless $\beta = \alpha$. Since

$$q(\beta)v = q(A)v = vw^T v = q(\alpha)v,$$

we see that $q(\beta) = q(\alpha)$. Therefore, a sufficient condition to force A to be a $p(t)$-cell would be that the only nonnegative integer solution of $q(t) = q(\alpha)$ is $t = \alpha$.

(b) For example, consider $p(t) = (t-4)(t-3)(t-2)$, and let $A = v = [\,1\,]$ and $w = [\,2\,]$. Here we have $\alpha = 4$ and $q(t) = (t-3)(t-2)$. Moreover, A is a nonnegative integer matrix such that $q(A) = [\,2\,] = vw^T$ and $q(\alpha) = 2 = w^T v$. However, the dominant eigenvalue of A is not α, and A is not a $p(t)$-cell.

Example 2.7. For the polynomial $p(t) = (t-3)(t^3 - t)$ in Example 2.4, we have $\alpha = 3$ and $q(t) = t^3 - t$, so Theorem 2.5 implies that for every nonnegative integer $p(t)$-cell A with dominant eigenvalue 3, there is a pair of positive integer vectors v, w such that

$$A^3 - A = vw^T \quad \text{and} \quad w^T v = q(3) = 24. \tag{2.6}$$

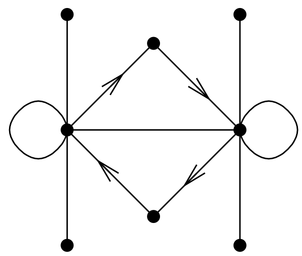

FIGURE 1. A $(t-3)(t^3-t)$-cell.

In particular, all such cells are of size $\leqslant 24$. Conversely, it is easy to see that $t = 3$ is the only real solution of $q(t) = 24$, so by the reasoning in Remark 2.6, every nonnegative integer matrix A satisfying (2.6) for some pair v, w is necessarily a $p(t)$-cell with dominant eigenvalue 3. Note also that the only $p(t)$-cells with a dominant eigenvalue less than 3 are also $q(t)$-cells; the bound in Theorem 2.3 shows that they are of size at most 2.

The largest $p(t)$-cell we have constructed is a 0, 1-matrix of size 8 whose support graph is displayed in Figure 1. Here, non-loop edges with no orientation represent pairs of edges in both directions.

As introduced by Duval [**D**], a *directed strongly regular graph* (DSRG) with parameters (k, λ, μ, ν) is a simple k-regular graph such that the number of directed paths of length 2 from i to j is either ν (if $i = j$), λ (if there is an edge $i \to j$) or μ (otherwise). Equivalently, these are the support graphs of the 0, 1-matrices A with diagonal 0 that satisfy

$$A^2 = \nu I + \lambda A + \mu(J - I - A), \quad AJ = JA = kJ. \tag{2.7}$$

The more familiar class of (undirected) strongly regular graphs consists of those DSRGs such that A is symmetric. These necessarily have $k = \nu$. Conversely, a DSRG with $k = \nu$ must have a symmetric adjacency matrix.

The following result shows that in some cases, the bound in Theorem 2.5 is sharp.

PROPOSITION 2.8. *Let $p(t) = (t-k)q(t)$ and $q(t) = t^2 + (1-\lambda)t + (1-\nu)$. If there exists a DSRG with parameters $(k, \lambda, 1, \nu)$ (i.e., $\mu = 1$), then it is a $p(t)$-cell of size $q(k)$. Moreover, this is the maximum possible size among all nonnegative integer $p(t)$-cells.*

Proof. Let A be (the adjacency matrix of) a DSRG with parameters $(k, \lambda, 1, \nu)$.

By rearranging (2.7) and setting $\mu = 1$, we obtain

$$q(A) = A^2 + (1-\lambda)A + (1-\nu)I = J,$$

so A is strongly connected. The k-regularity also implies $(A - kI)q(A) = (A - kI)J = 0$, so A is a $p(t)$-cell. Furthermore,

$$J^2 = q(A)J = q(k)J,$$

so $q(k)$ is the size of A. In particular, $q(k) \neq 0$, so k is a simple root of $p(t)$ and Theorem 2.5 implies that $q(k)$ is the maximum possible size of any nonnegative integer $p(t)$-cell with dominant eigenvalue k. If a nonnegative integer $p(t)$-cell has a dominant eigenvalue less than k, then it must be a $q(t)$-cell. Since $q(t)$ is quadratic, Theorem 2.3 implies that the size of such a cell must be $\leqslant 1 + \alpha$ where $\alpha < k$ is the largest root of $q(t)$. Since k is the number of 1's in each row of A, we have $k + 1 \leqslant q(k)$, so all nonnegative integer $p(t)$-cells are of size at most $q(k)$. $\square$

Examples of DSRG parameter sets with $\mu = 1$ include the symmetric cases $(2, 0, 1, 2)$ (a 5-cycle), $(3, 0, 1, 3)$ (the Petersen graph), and $(7, 0, 1, 7)$ (the Hoffman-Singleton graph). For further examples, including nonsymmetric cases (i.e., $k > \nu$), see [**D**] and the tables maintained by A. E. Brouwer [**B**].

3. Algebraic cells

Let $\mathcal{A}$ be a finite-dimensional associative algebra over a field $\mathbf{k}$ of characteristic 0.

In the following, we will work in the category of $\mathcal{A}$-Modules-With-Basis. The objects of this category are pairs (M, B) consisting of a finite-dimensional left $\mathcal{A}$-module M and a $\mathbf{k}$-basis $B \subset M$. A morphism $(M, B) \to (M', B')$ is an $\mathcal{A}$-module homomorphism $M \to M'$ that restricts to a map $B \to B'$.

Naturally associated to any module-with-basis (M, B) is a directed graph $G(M, B)$ with vertex set B and an edge $b' \leftarrow b$ for each pair $b, b' \in B$ such that the coefficient of b' in ab is nonzero for some element $a \in \mathcal{A}$. In these terms, a subobject (M', B') of (M, B) in the $\mathcal{A}$-Modules-With-Basis category consists of a $\mathbf{k}$-subspace M' of M spanned by an outward-closed subset $B' \subset B$ (i.e., $b \in B'$, $b' \leftarrow b$ implies $b' \in B'$). Note that (M, B) is irreducible in this category precisely if the graph $G(M, B)$ is strongly connected; however, in such cases M need not be irreducible in the category of $\mathcal{A}$-modules. We will refer to the irreducible modules-with-basis as *$\mathcal{A}$-cells*.

Now fix a finite generating set $S = \{a_1, \ldots, a_l\}$ for $\mathcal{A}$.

For each module-with-basis (M, B), define $G_S(M, B)$ to be the subgraph of $G(M, B)$ in which there is an edge $b' \leftarrow b$ only if the coefficient of b' in $a_i b$ is nonzero for some generator a_i. It is easy to see that if $b' \leftarrow b$ in $G(M, B)$ then there is a directed path from b to b' in $G_S(M, B)$, so these graphs have the same strongly connected components, and (M, B) is an $\mathcal{A}$-cell if and only if $G_S(M, B)$ is strongly connected.

As an example, note that the $p(t)$-cells discussed in the previous section are the $\mathcal{A}$-cells for the algebra $\mathcal{A} = \mathbb{Q}[t]/(p(t))$.

We will say that a module-with-basis (M, B) is nonnegative (respectively, integral) with respect to S if the matrix entries representing the action of S on (M, B) (i.e., the coefficients of b' in $a_i b$ for all i and all $b, b' \in B$) are nonnegative (respectively, integral).

Theorem 3.1. *For any finite-dimensional $\mathbf{k}$-algebra $\mathcal{A}$ with generating set S, there are only finitely many $\mathcal{A}$-cells that are nonnegative and integral with respect to S.*

Proof. Let $a = a_1 + \cdots + a_l \in \mathcal{A}$. Since $\mathcal{A}$ is finite-dimensional, the powers of a must be linearly dependent, so there is a nonzero polynomial $p(t) \in \mathbf{k}[t]$ such that $p(a) = 0$.

Now consider an $\mathcal{A}$-cell (M, B) that is nonnegative and integral with respect to S, and let A denote the nonnegative integer matrix representing the action of a on (M, B). The nonnegativity implies that the support graph of A is precisely $G_S(M, B)$, so A is strongly connected and is therefore a $p(t)$-cell.

Although $p(t)$ need not have integer coefficients, the minimal polynomial of A necessarily does, and is a divisor of $p(t)$. Thus without loss of generality, we could replace $p(t)$ with the least common multiple of all monic integer polynomials that divide $p(t)$ in $\mathbf{k}[t]$. Applying Theorem 2.1, we conclude that there are only finitely many possible matrices A representing the action of a in a nonnegative integer $\mathcal{A}$-cell. Moreover, A itself is the sum of l nonnegative integer matrices representing the actions of each generator a_i, and these matrices completely determine the module-with-basis up to isomorphism. Since there are finitely many ways to decompose A into such a sum, the result follows. $\square$

4. Consequences for W-graphs

Let W be a finite Coxeter group with simple reflections $S = \{s_1, \ldots, s_l\}$, and $\mathcal{H}(W, S)$ the associated Iwahori-Hecke algebra over the rational function field $\mathbb{Q}(q^{1/2})$.

One knows that $\mathcal{H} = \mathcal{H}(W, S)$ may be presented as the algebra generated by a set of elements $T = \{T_1, \ldots, T_l\}$ satisfying the quadratic relations

$$(T_i - q)(T_i + 1) = 0$$

and the braid relations

$$\begin{aligned} (T_iT_j)^{m/2} &= (T_jT_i)^{m/2} && \text{if } m \text{ is even,} \\ T_i(T_jT_i)^{(m-1)/2} &= T_j(T_iT_j)^{(m-1)/2} && \text{if } m \text{ is odd,} \end{aligned}$$

for all distinct i, j, where $m = m_{ij}$ denotes the order of s_is_j in W.

Following Kazhdan and Lusztig [**KL**], an $\mathcal{H}$-module-with-basis (M, B) is a *W-graph* if there is a matrix $A = [a(u, v)]_{u,v \in B}$ of scalars and a subset $\tau(u) \subseteq \{1, 2, \ldots, l\}$ for each $u \in B$ such that the action of $\mathcal{H}$ on M has the form

$$T_i(v) = \begin{cases} -v & \text{if } i \in \tau(v), \\ qv + q^{1/2} \sum_{u:\, i \in \tau(u)} a(u, v)u & \text{if } i \notin \tau(v). \end{cases} \tag{4.1}$$

for all $v \in B$ and all generators T_i. More precisely, one should regard the combinatorial datum (A, τ) as the W-graph, and (M, B) as the $\mathcal{H}$-module-with-basis encoded by it.

It is easy to show that any operator T_i having the form of (4.1) automatically satisfies the quadratic relation $(T_i - q)(T_i + 1) = 0$, so verifying that the pair (A, τ) is a W-graph amounts to checking that these operators satisfy the braid relations.

The motivation for studying W-graphs, as shown by Kazhdan and Lusztig, is that the left action of the generators T_i on the Kazhdan-Lusztig basis $\{C_w : w \in W\}$ of $\mathcal{H}$ has the form of a W-graph. Indeed, letting $\ell(\cdot)$ denote the length function on W, the associated matrix A turns out to be $[\mu(u,v) + \mu(v,u)]_{u,v\in W}$, where $\mu(u,v)$ denotes the coefficient of $q^{(\ell(v)-\ell(u)-1)/2}$ in the Kazhdan-Lusztig polynomial $P_{u,v}(q)$, and $\tau(u) = \{i : \ell(s_i u) < \ell(u)\}$ (i.e., the left descent set of u).

We prefer the convention that all W-graphs (A,τ) must be *reduced* in the sense that

$$\tau(u) \subseteq \tau(v) \;\Rightarrow\; a(u,v) = 0.$$

Although the Kazhdan-Lusztig W-graph (for example) is not reduced, this causes no loss of generality. Indeed, whenever $\tau(u) \subseteq \tau(v)$, one sees from (4.1) that $T_i(v)$ does not depend on $a(u,v)$, so redefining $a(u,v) = 0$ in this situation has no effect on the module.

In this way, reduced W-graphs (A,τ) are as sparse as possible, and it is not hard to see that the support graph of A coincides with the loop-deleted part of the graph $G_T(M,B)$ if and only if (A,τ) is reduced. We define (A,τ) to be a *W-cell* if (A,τ) is reduced and A is strongly connected. Thus $\mathcal{H}$-cells are the modules-with-basis arising from W-cells.

As part of a project to understand the essential combinatorial features of the Kazhdan-Lusztig W-graph, we introduced in [**S**] the notion of an *admissible* W-graph. (The matrices we used in [**S**] are transposed from those in (4.1), but this has no substantial effect on what constitutes a W-graph, or admissibility.) These are the W-graphs (A,τ) such that

(i) A has nonnegative integer entries,
(ii) the support graph of A is bipartite, and
(iii) $a(u,v) = a(v,u)$ unless $\tau(u) \subseteq \tau(v)$ or $\tau(v) \subseteq \tau(u)$.

Given that the Kazhdan-Lusztig polynomials $P_{u,v}(q)$ for finite W are known to have nonnegative integer coefficients, it is not hard to see that the Kazhdan-Lusztig W-graph and all of its cells are admissible.

The following result resolves Question 2.5 in [**S**].

THEOREM 4.1. *For a finite Coxeter group W, there are only finitely many W-cells over the nonnegative integers. In particular, there are only finitely many admissible W-cells.*

Proof. Let (A,τ) be a W-cell and assume A has nonnegative integer entries. Although the action of T_i in (4.1) is neither nonnegative nor integral in that case, if we set $q = 1$ and add the identity operator, we obtain operators E_i such that

$$E_i(v) = \begin{cases} 0 & \text{if } i \in \tau(v), \\ 2v + \sum_{u:\, i\in\tau(u)} a(u,v)u & \text{if } i \notin \tau(v). \end{cases} \tag{4.2}$$

Moreover, setting $q = 1$ in the defining relations for $\mathcal{H}$ yields relations that define the group algebra $\mathbb{Q}W$. Thus the operators $\{E_1, \ldots, E_l\}$ satisfy every relation satisfied by the elements $X = \{1+s_1, \ldots, 1+s_l\}$ in $\mathbb{Q}W$. In this way, (4.2) defines a cell for the subalgebra $\mathcal{A}$ of $\mathbb{Q}W$ generated by X, and it is nonnegative and integral with respect to X.

By Theorem 3.1, we know that there are only finitely many such $\mathcal{A}$-cells. Conversely, an $\mathcal{A}$-cell might not arise from a W-cell (A, τ) as in (4.2), but if it does, we claim that one may recover A and τ. Indeed, we must have $\tau(v) = \{i : E_i(v) = 0\}$. Now having recovered all τ values, $a(u, v)$ must either be the coefficient of u in $E_i(v)$ for an index $i \in \tau(u) \setminus \tau(v)$, or if there is no such index, zero. (The latter follows since A must be reduced.) $\square$

As a final remark, we mention that it would be interesting to have an effective algorithm for constructing the nonnegative integer $p(t)$-cells for any given $p(t) \in \mathbb{Z}[t]$. Although it is unlikely that any such algorithm would be practical for a direct construction of admissible W-cells along the lines of the above proof, it turns out that there are potential indirect applications for such an algorithm toward W-cell construction. For example, in the case of the Weyl group F_4, there is a family of admissible F_4-cells (including all of the Kazhdan-Lusztig cells belonging to the largest two-sided cell) each of which has a pair of canonically associated nonnegative integer $p(t)$-cells for the polynomial $p(t) = (t-3)(t^3-t)$ discussed in Examples 2.4 and 2.7. The largest $p(t)$-cell that arises in this way is the one of order 8 displayed in Figure 1.

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