Folding by Automorphisms

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Let Φ be a simply-laced root system with simple roots $\Delta = \{\alpha_i : i \in I\}$ embedded in the real vector space V with inner product $\langle\cdot,\cdot\rangle$. For convenience, we may assume that the roots have been normalized so that $\langle\alpha,\alpha\rangle = 2$ for all $\alpha \in \Phi$. As a second convenience, we may assume that V has been enlarged, if necessary, so that $\langle\cdot,\cdot\rangle$ is nondegenerate on V.

Now let σ be a diagram automorphism of (Φ, Δ). Given our choice of normalization, this amounts to a permutation of I such that

$$\langle\alpha_{\sigma(i)}, \alpha_{\sigma(j)}\rangle = \langle\alpha_i, \alpha_j\rangle \quad (i, j \in I).$$

In particular, if we extend the map $\alpha_i \mapsto \alpha_{\sigma(i)}$ linearly, we may view σ as an isometry of $\operatorname{Span}\Delta$. We may extend σ further to an isometry defined on all of V by insisting that it acts trivially on the orthogonal complement of Δ (here we are using nondegeneracy).

Note that $\sigma s_i \sigma^{-1} = s_{\sigma(i)}$, so σ acts via conjugation as an automorphism $w \mapsto w^\sigma$ of the Coxeter group $W = W(\Phi, \Delta)$.

We now introduce the key extra condition:

$$\textit{simple roots in the same } \sigma\textit{-orbit must be orthogonal.} \tag{1}$$

Equivalently, σ-orbits are independent (i.e., edge-free) sets in the Dynkin diagram.

Let $I^\sigma = \{B_1, \dots, B_l\}$ denote the set of σ-orbits on I; this is a partition of I into disjoint blocks of the form $\{\sigma^k(i) : k \in \mathbb{Z}\}$ for various $i \in I$. For each block B_j, we define

$$\beta_j = \sum_{i \in B_j} \alpha_i.$$

Note that each β_j has squared length $2b_j$, where $b_j := |B_j|$. Moreover,

$$\langle\beta_i, \beta_j^\vee\rangle = -N_{ij}/b_j, \tag{2}$$

where N_{ij} denotes the number of edges in the Dynkin diagram between block B_i and block B_j (assuming $i \neq j$). Since σ acts transitively on each block, it follows each of the b_j nodes in B_j have the same the same number of neighbors in B_i. Thus for $i \neq j$,

$$-\langle\beta_i, \beta_j^\vee\rangle = \#\text{ of nodes in } B_i \text{ adjacent to any fixed member of } B_j. \tag{3}$$

In particular, $\Delta^\sigma := \{\beta_1, \dots, \beta_l\}$ forms a set of simple roots for some crystallographic (but probably not simply-laced) root system Φ^σ in V.

The two main issues at this point are: (1) how to relate Φ and Φ^σ, and (2) the inverse problem; i.e., given a multiply-laced crystallographic root system, can we realize it as a "folding" $(\Phi^\sigma, \Delta^\sigma)$ of a simply-laced root system (Φ, Δ) by some automorphism σ?

CLAIM 1. *If β is a sum of pairwise orthogonal roots in Φ comprising a single σ-orbit, then β is a root in Φ^σ. Conversely, all roots in Φ^σ have this form.*

REMARK 2. The orthogonality constraint is necessary. For example, label the simple roots α_i of $\mathrm{Aff}(A_3)$ so that the 4-cycle $0 \to 1 \to 2 \to 3 \to 0$ is a diagram automorphism. Although this automorphism does not satisfy (1), the automorphism $0 \leftrightarrow 2, 1 \leftrightarrow 3$ does. Moreover, $s_0\alpha_1 = \alpha_0 + \alpha_1$ is clearly a root, and $\{\alpha_0 + \alpha_1, \alpha_2 + \alpha_3\}$ is its orbit. However, the two roots in this orbit have inner product -2 (not zero), and their sum has squared length 0, and hence cannot be a root of the folded root system (which in this case happens to be isomorphic to $\mathrm{Aff}(A_1)$).

For each block B_j, let t_j denote the reflection corresponding to the simple root $\beta_j \in \Phi^\sigma$, and define

$$\bar{s}_j := \prod_{i \in B_j} s_i,$$

a product of commuting reflections in W. Note that $\bar{s}_j$ is fixed under conjugation by σ.

CLAIM 3. *The map $t_j \mapsto \bar{s}_j$ extends to an isomorphism from the Coxeter group $W(\Phi^\sigma, \Delta^\sigma)$ to W^σ, the subgroup of W fixed by σ.*

Proof. Consider the reflection actions of $W(\Phi^\sigma, \Delta^\sigma)$ and W on V. A simple calculation shows that $\bar{s}_j(\alpha_i) - \alpha_i$ is the sum of all simple roots indexed by nodes in B_j that are adjacent to node i in the Dynkin diagram. It follows that

$$\bar{s}_j(\beta_i) = \beta_i + \sum_{k \in B_j} n_k \alpha_k,$$

where n_k is the number of nodes in B_i adjacent to node k. Recalling from (3) that n_k is the constant $-\langle \beta_i, \beta_j^\vee \rangle$ (independent of k), we obtain that $\bar{s}_j(\beta_i) = \beta_i - \langle \beta_i, \beta_j^\vee \rangle \beta_j$. In other words, the actions of $\bar{s}_j$ and t_j on the span of Δ^σ are identical. Since the reflection representation of any Coxeter group is faithful, it follows that the map $t_j \mapsto \bar{s}_j$ extends to an injective group homomorphism $W(\Phi^\sigma, \Delta^\sigma) \to W$.

To complete the proof, we argue that every $w \in W$ fixed by σ is in the subgroup generated by $\{\bar{s}_j : 1 \leqslant j \leqslant l\}$. Proceeding by induction with respect to length, there is nothing to prove if $\ell(w) = 0$. Otherwise, there is a simple reflection s_i such that $\ell(ws_i) < \ell(w)$. Since σ is length-preserving and fixes w, it follows that the same is true for every simple reflection in the σ-orbit of s_i. Thus w is a longest coset representative relative to the parabolic subgroup indexed by some block B_j. However, (1) implies that $\bar{s}_j$ is the longest element of this parabolic subgroup, and hence $\ell(w\bar{s}_j) = \ell(w) - |B_j| < \ell(w)$. Applying the induction hypothesis to $w\bar{s}_j$ completes the proof. □

Proof of Claim 1. By Claim 3, the actions of W^σ and $W(\Phi^\sigma, \Delta^\sigma)$ on the span of Δ^σ are naturally isomorphic, so every root in Φ^σ has the form $w\beta_j$, where $w \in W^\sigma$, $\beta_j \in \Delta^\sigma$. Furthermore, it is clear (from the definition) that β_j is the sum of the roots in some pairwise orthogonal σ-orbit on Φ. However, every $w \in W^\sigma$ permutes the set of σ-orbits of roots in Φ, and since W^σ acts as a group of isometries, it also permutes those orbits whose members are pairwise orthogonal. Thus every root in Φ^σ has the claimed form.

Conversely, let $\{\gamma_1, \ldots, \gamma_k\} \subset \Phi$ be an orthogonal σ-orbit with sum $\gamma = \gamma_1 + \cdots + \gamma_k$. We seek to show that $\gamma \in \Phi^\sigma$. Without loss of generality, we may assume that the roots γ_i are positive (σ permutes the positive and negative roots separately) and proceed by induction with respect to the height of γ. Given that the roots are orthogonal, we have

$$\langle \gamma, \gamma \rangle = \sum_{i=1}^{k} \langle \gamma_i, \gamma_i \rangle = 2k > 0. \tag{4}$$

Thus there is a simple root α_i such that $\langle \gamma, \alpha_i \rangle > 0$. Since σ fixes γ, it follows that the same is true for all simple roots in the σ-orbit of α_i, and thus $\langle \gamma, \beta_j \rangle > 0$ for some j. Note that the $\bar{s}_j$-image of $\{\gamma_1, \ldots, \gamma_k\}$ is another orthogonal σ-orbit, but the action of $\bar{s}_j$ (or t_j) on γ subtracts a positive multiple of β_j, so it is an orbit of lower height. This completes the induction, aside from the possibility that this lower orbit consists only of negative roots. However, the only positive roots of Φ sent to negative roots by $\bar{s}_j$ are the simple roots indexed by B_j. Hence this exceptional case occurs only when $\gamma = \beta_j$. □

In proving that all roots in Φ^σ are orthogonal orbit-sums, the only use of orthogonality is in (4), where we needed it to guarantee that an orbit-sum γ had positive squared-length. Thus if $\langle \cdot, \cdot \rangle$ happens to be positive definite (i.e., if Φ is finite), then orthogonality is no longer a necessary assumption. In other words, we have

Claim 4. *If Φ is finite, then Φ^σ consists of all sums of roots in individual σ-orbits. In particular, all such orbits consist of pairwise orthogonal roots.*

We now turn to the inverse problem. Let $A = [a_{ij}]_{1 \leqslant i,j \leqslant l}$ be the Cartan matrix of a crystallographic root system of rank l. We seek to realize this root system as a folding $(\Phi^\sigma, \Delta^\sigma)$ of some simply-laced root system (Φ, Δ) by an automorphism σ.

To begin, let $\beta_1, \ldots, \beta_l$ denote the simple roots of the (as yet unknown) root system, and let $b_1, \ldots, b_l$ denote scalars (also unknown) such that $\langle \beta_i, \beta_i \rangle = 2b_i$. Thus we have

$$\langle \beta_i, \beta_j^\vee \rangle = a_{ij} \quad (1 \leqslant i, j \leqslant l),$$

and hence $b_i/b_j = a_{ij}/a_{ji}$ (if $a_{ji} \neq 0$). It follows that the relative length of all roots in each irreducible component are determined by A. If we arbitrarily set $b_i = 1$ for one node i from each irreducible component and then rescale as necessary, we may assume that

(A1) $b_1, \dots, b_l$ are positive integers, and
(A2) $b_i \geqslant -a_{ij}$ for all $j \neq i$.

Now set $n = b_1 + \cdots + b_l$ and arbitrarily partition $I = \{1, \dots, n\}$ into blocks B_i of size b_i. Fix an arbitrary permutation σ of I whose orbits (i.e., cycles) are $B_1, \dots, B_l$. We claim that one may construct a simply-laced Dynkin diagram Γ on the vertex set $[n]$ such that

(A3) σ is an automorphism of Γ,
(A4) there are no edges internal to any block B_i, and
(A5) the number of edges between B_i and B_j is $N_{ij} := -b_j a_{ij} = -b_i a_{ji}$ (for $i \neq j$).

To prove this claim, consider that the action of σ on $B_i \times B_j$ consists of $\gcd(b_i, b_j)$ cycles of length $\operatorname{lcm}(b_i, b_j)$. Since N_{ij} is evidently a multiple of both b_i and b_j, it is therefore a multiple of the cardinality of σ-orbits on $B_i \times B_j$. Furthermore, this multiple does not exceed the total number of orbits available, by (A2). Thus we may arbitrarily select $N_{ij}/\operatorname{lcm}(b_i, b_j)$ σ-orbits from $B_i \times B_j$ as edges for the graph Γ.

Now let (Φ, Δ) be the simply-laced root system with Dynkin diagram Γ. If we fold this root system by σ, one sees by comparing (A5) and (2) that the folded root system will have Cartan matrix A. This proves

Claim 5. *Every crystallographic root system may be realized as a folding* $(\Phi^\sigma, \Delta^\sigma)$ *of a simply-laced root system* (Φ, Δ) *by some diagram automorphism* σ.