Kostka-Foulkes Polynomials of General Type
Generalized Kostka Polynomials Workshop
American Institute of Mathematics, 18–22 July 2005

John R. Stembridge ⟨jrs@umich.edu⟩

Department of Mathematics
University of Michigan
Ann Arbor, Michigan 48109–1043

Contents

These lecture notes are aimed at someone who has read Chapter III of Macdonald's book [**M2**] and wants to know how it generalizes to finite root systems. Or at least, how the parts most relevant to the Kostka-Foulkes story generalize. One should keep in mind that Macdonald's Chapter III is effectively about $\mathcal{A}_\infty$ (or $GL(\infty)$), whereas we will be talking about finite-dimensional root systems and groups, and this finite-vs.-infinite discrepancy (as well as the SL vs. GL issue) means that this won't be an exact generalization.

Our plan is to discuss four definitions of (or contexts for) the Kostka-Foulkes polynomials, and why they are equivalent.

1. Hall-Littlewood-Macdonald Polynomials

Before getting down to business, we need to introduce some basic notation.

Let Φ be a finite, reduced crystallographic root system embedded in a real Euclidean space with inner product $\langle \cdot, \cdot \rangle$. We let $\alpha^\vee = 2\alpha/\langle \alpha, \alpha \rangle$ denote the co-root corresponding to a root α, and Φ^+ a set of positive roots.

We let Λ denote any lattice that is compatible with Φ; i.e., it should contain Φ and satisfy the condition that $\langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}$ for all $\lambda \in \Lambda$ and $\alpha \in \Phi$. Members of Λ are called (integral) weights.

A weight λ is dominant if $\langle \lambda, \alpha^\vee \rangle \geqslant 0$ for all $\alpha \in \Phi^+$; we let Λ^+ denote the dominant part of Λ. (Beware that the superscripts on Φ^+ and Λ^+ refer to two separate notions of positivity that are dual to each other.)

The Weyl group W is generated by reflections through the hyperplanes orthogonal to the roots. It permutes Λ and Φ. We partially order Λ via the rule

$$\lambda \geqslant \mu \;\Leftrightarrow\; \lambda - \mu \in \mathbb{N}\Phi^+.$$

The (unique) dominant member of each W-orbit in Λ is a $\geqslant$-maximum within its orbit.

For example, in the standard coordinates for $\Phi = \mathcal{A}_{n-1}$, we can take $\Lambda = \mathbb{Z}^n$. Here, the dominant vectors have decreasing coordinates—these would be partitions with at most n parts if you allow negative parts, and the partial ordering is majorization (or dominance).

Now let $\mathbb{Z}[\Lambda] = \mathbb{Z}\{e^\lambda : \lambda \in \Lambda\}$ denote the group ring of Λ. This is isomorphic to a ring of Laurent polynomials in n variables, where $n = \operatorname{rk}\Lambda$. The W-action on Λ extends to a W-action on $\mathbb{Z}[\Lambda]$, and we let $J(\cdot)$ denote the skew-symmetrizing operator; i.e.,

$$J(f) = \sum_{w\in W} \operatorname{sgn}(w) w.f \qquad (f \in \mathbb{Z}[\Lambda]).$$

Letting ρ denote half the sum of the positive roots,

$$\chi(\lambda) := J(e^{\lambda+\rho})/J(e^\rho) \qquad (\lambda \in \Lambda^+)$$

is the Weyl character indexed by λ. It is the character of an irreducible $\mathfrak{g}$-module of highest weight λ, where $\mathfrak{g}$ is a semisimple Lie algebra with root system Φ. As a ratio of two skew-symmetric objects, it is clear that $\chi(\lambda)$ is W-symmetric, and it is not hard to show that the Weyl characters form a free $\mathbb{Z}$-basis of $\mathbb{Z}[\Lambda]^W$. (In particular, every skew-symmetric object is divisible by $J(e^\rho)$.) Weyl characters are the generalizations of Schur functions to arbitrary root systems, and it is easy to recognize the above definition as a generalization of the bi-alternant formula.

Now we can introduce the generalization of Hall-Littlewood polynomials to all root systems. These were first studied by Macdonald in connection with the theory of p-adic groups (more about this later, but see [**M1**]). It would be fitting to call these "Macdonald polynomials," but everyone at this workshop would agree that this is not such a great idea. As a compromise, we call them Hall-Littlewood-Macdonald (HLM) polynomials.

Definition. For each $\lambda \in \Lambda^+$, let

$$P(\lambda;t) := J\Big(e^{\lambda+\rho}\textstyle\prod_{\alpha\in\Phi:\langle\lambda,\alpha\rangle>0}(1-te^{-\alpha})\Big)/J(e^\rho).$$

Notice that we silently enlarged the ground ring to $\mathbb{Z}[t]$.

Remark. As far as we know, the above definition is new. Certainly it looks different from the definition you get by extrapolating from Macdonald's Chapter III:

$$P(\lambda;t) = W_\lambda(t)^{-1} \cdot J\Big(e^{\lambda+\rho}\textstyle\prod_{\alpha>0}(1-te^{-\alpha})\Big)/J(e^\rho), \tag{1}$$

where W_λ denotes the W-stabilizer of λ, and

$$W_\lambda(t) := \sum_{w \in W_\lambda} t^{\ell(w)}$$

is its Poincaré series. We prefer not to use (1) as the definition, since it does not as readily reveal that $P(\lambda;t)$ has coefficients in $\mathbb{Z}[t]$. One would (incorrectly) expect to need $\mathbb{Q}(t)$.

Here are several basic properties of HLM polynomials.

(i) *$P(\lambda;t)$ is W-invariant and has $\mathbb{Z}[t]$-coefficients.*

Proof. This is obvious from the definition. $\square$

(ii) *$P(\lambda;t) = \chi(\lambda) +$ lower terms with respect to $\geqslant$.*

Hence $\{P(\lambda;t) : \lambda \in \Lambda^+\}$ is a free $\mathbb{Z}[t]$-basis of $\mathbb{Z}[t][\Lambda]^W$.

Proof. Recalling the definition of ρ, we have

$$J(e^\rho)P(\lambda;t) = \sum_{w \in W} \operatorname{sgn}(w) e^{w\lambda} \cdot \prod_{\langle\lambda,\alpha\rangle=0} e^{w\alpha/2} \cdot \prod_{\langle\lambda,\alpha\rangle>0} (e^{w\alpha/2} - te^{-w\alpha/2}).$$

Each term $\pm t^l e^{\mu+\rho}$ in this expansion satisfies $\mu + \rho \leqslant w\lambda + \rho \leqslant \lambda + \rho$ for some $w \in W$, so if μ is dominant and $\chi(\mu)$ occurs in the expansion of $P(\lambda;t)$, we must have $\mu \leqslant \lambda$. If equality occurs, then $w\lambda = \lambda$ (i.e., $w \in W_\lambda$), and each time we select $w\alpha$ or $-w\alpha$, we must take a positive root. The latter implies that if $\langle\lambda,\alpha\rangle = 0$, then $w\alpha > 0$, since the above expansion doesn't allow for choosing $-w\alpha$ in such cases. However, W_λ is the Weyl group of the root system $\{\alpha \in \Phi : \langle\lambda,\alpha\rangle = 0\}$, and every nontrivial element of a Weyl group must send one of its positive roots to a negative root, so only the $w = 1$ term contributes to the coefficient of $\chi(\lambda)$ in $P(\lambda;t)$, and it is easy to see that this contribution is 1. $\square$

(iii) *$P(\lambda;0) = \chi(\lambda)$.*

Proof. Another one straight from the definition. $\square$

(iv) *$P(\lambda;1) = m(\lambda) := |W_\lambda|^{-1} \sum_{w \in W} e^{w\lambda}$.*

The $m(\lambda)$'s are orbit sums that generalize the monomial symmetric functions.

Proof. The Weyl Denominator Formula $J(e^\rho) = e^\rho \prod_{\alpha>0}(1 - e^{-\alpha})$ implies

$$P(\lambda;1) = \sum_{w \in W} w\Big(e^\lambda \prod_{\langle\lambda,\alpha\rangle=0} \frac{1}{1-e^{-\alpha}}\Big) = \sum_{x \in W^\lambda} x\Big(e^\lambda \cdot \sum_{w \in W_\lambda} w\Big(\prod_{\langle\lambda,\alpha\rangle=0} \frac{1}{1-e^{-\alpha}}\Big)\Big),$$

where W^λ denotes any set of coset representatives for W/W_λ. In particular, notice that the inner sum is recognizable as the W_λ-version of $P(0;t)$ at $t = 1$. But this particular case is trivial: (ii) implies $P(0;t) = \chi(0) = 1$, since there are no "lower terms" in this case. Hence, the inner sum is 1 and $P(\lambda;1) = \sum_{x \in W^\lambda} e^{x\lambda} = m(\lambda)$. $\square$

Definition. *The Kostka-Foulkes polynomials for the root system Φ are the coefficients $K_{\lambda,\mu}(t) \in \mathbb{Z}[t]$ in the expansion*

$$\chi(\lambda) = \sum_{\mu \in \Lambda^+} K_{\lambda,\mu}(t) P(\mu; t) \qquad (\lambda \in \Lambda^+).$$

Note that (ii) implies that $K_{\lambda,\mu}(t) = 0$ unless $\lambda \geqslant \mu$.

Also, (iii) implies that $K_{\lambda,\mu}(1)$ is the coefficient of e^μ in $\chi(\lambda)$; i.e., dimension of the μ weight space of an irreducible $\mathfrak{g}$-module of highest weight λ. Thus it is nonnegative. Alternatively, Garsia has pointed out that we can derive the Freudenthal formula for weight multiplicity, and hence the positivity of $K_{\lambda,\mu}(1)$, directly from the definition of $\chi(\lambda)$.

Remark. Macdonald proved that the HLM polynomials encode "spherical functions" for p-adic groups, and Chapter V of Macdonald's book tells the $GL(n)$ version of this story. Roughly speaking, if G is a p-adic Chevalley group whose root system and weight lattice are dual to Φ and Λ, and K is a maximal compact subgroup of G, then the Hecke algebra $H(G, K)$ of (K, K)-bi-invariant $\mathbb{Q}$-valued functions on G under convolution is isomorphic to $\mathbb{Q}[\Lambda]^W$. In particular, $H(G, K)$ is commutative and (G, K) is a Gelfand pair.

Furthermore, the double cosets $K\backslash G/K$ are indexed by Λ^+, and under the isomorphism, the HLM polynomial $P(\lambda; t)$ specialized to $t = 1/p$ corresponds to a bi-invariant function that vanishes everywhere except on the double coset indexed by λ.

This will come back in a big way in §4.

2. Kostka-Foulkes Polynomials and Root Partitions

Define a graded version of the Kostant partition function via the formal expansion

$$\prod_{\alpha \in \Phi^+} \frac{1}{1 - te^\alpha} = \sum_{\gamma \in \mathbb{N}\Phi^+} \mathcal{P}(\gamma; t) e^\gamma.$$

Note that the coefficient of t^l in $\mathcal{P}(\gamma; t)$ is the number of ways to partition γ into an unordered sum of l positive roots.

The cone $\mathbb{N}\Phi^+$ is simplicial, being generated by the simple roots, and it is not hard to show that $\langle \alpha, \rho^\vee \rangle = 1$ for all simple roots α, where $\rho^\vee$ denotes half the sum of the positive co-roots. Thus $\mathcal{P}(\gamma; t)$ (for $\gamma \in \mathbb{N}\Phi^+$) is monic of degree $\langle \gamma, \rho^\vee \rangle$, the height of γ.

The main goal of this section is to prove the following formula for the Kostka-Foulkes polynomials due originally to Kato [**K**]:

$$K_{\lambda,\mu}(t) = \sum_{w \in W} \operatorname{sgn}(w) \mathcal{P}(w(\lambda + \rho) - (\mu + \rho); t). \tag{2}$$

Since the height of $w(\lambda + \rho)$ is maximized at $w = 1$, it follows immediately that $K_{\lambda,\mu}(t)$ is monic of degree $\langle \lambda - \mu, \rho^\vee \rangle$.

As an aside, we would like to mention an interesting research topic related to this formula. Let Ψ be (potentially) any finite subset or multisubset of Λ, and define a graded partition function $\mathcal{P}_\Psi$ via the expansion

$$\prod_{\nu\in\Psi}\frac{1}{1-te^\nu}=\sum_{\gamma\in\Lambda}\mathcal{P}_\Psi(\gamma;t)e^\gamma.$$

Note that in general, $\mathcal{P}_\Psi(\gamma;t)$ is a formal series in $\mathbb{Z}[[t]]$, not necessarily a polynomial. From the general theory of counting lattice points in polytopes (e.g., see §4.6 of [**EC1**]), it follows that $\mathcal{P}_\Psi(\gamma;t)$ is always a rational function of t with poles at roots of unity.

We now introduce

$$K^\Psi_{\lambda,\mu}(t):=\sum_{w\in W}\operatorname{sgn}(w)\mathcal{P}_\Psi(w(\lambda+\rho)-(\mu+\rho);t).$$

Bearing in mind (2), these are generalizations of the Kostka-Foulkes polynomials.

Problem. *Identify reasonable sufficient conditions on Ψ such that $K^\Psi_{\lambda,\mu}(t)$ has nonnegative coefficients for all dominant $\lambda,\mu\in\Lambda$.*

A reasonable (but false) conjecture is that $F_\Psi=\sum_{\nu\in\Psi}e^\nu$ should be the character of a $\mathfrak{b}$-submodule of a $\mathfrak{g}$-module, where $\mathfrak{b}$ denotes a Borel subalgebra of $\mathfrak{g}$. This is true for $\mathcal{A}_1$ (i.e., $K^\Psi_{\lambda,\mu}(t)$ has nonnegative coefficients in such cases), but there are counterexamples available in $\mathcal{A}_2$. In any case, we expect that there should be a nonnegativity theorem for the multisets of weights Ψ that occur in some large class of $\mathfrak{b}$-submodules of $\mathfrak{g}$-modules.

R. Brylinski's proof of (2) (see [**B1**]).

For each dominant $\mu\in\Lambda$, let us define

$$Q(\mu;t):=J\Big(e^{\mu+\rho}\textstyle\prod_{\alpha>0}(1-te^\alpha)^{-1}\Big)/J(e^\rho).$$

This should be interpreted as a W-invariant element of $\mathbb{Z}[\Lambda][[t]]$; i.e., the ring of formal series in t over the ground ring $\mathbb{Z}[\Lambda]$, not the (smaller) group ring of Λ over $\mathbb{Z}[[t]]$.

Beware that these $Q(\mu;t)$ are *not* direct generalizations of the Q-function one finds in Chapter III of Macdonald's book. Those Q-functions are $\mathbb{Z}[t]$-multiples of P-functions, whereas these objects are infinite series. Nevertheless, both flavors of Q-functions play a similar role as duals of P-functions.

Here are several basic properties of the $Q(\mu;t)$'s.

(v) $Q(\mu;t)=\chi(\mu)+$ *higher terms with respect to* $\geqslant$.

Proof. For each term $t^le^{\lambda+\rho}$ in the expansion of $e^{\mu+\rho}\prod(1-te^\alpha)^{-1}$, we have $\lambda\geqslant\mu$. Furthermore, equality occurs only if $l=0$. $\square$

(vi) $Q(\mu;t) = W_\mu(t)\Theta(t)P(\mu;t)$, *where* $\Theta(t) := \displaystyle\prod_{\alpha\in\Phi} \frac{1}{1-te^\alpha}$.

Proof. Since $1/\Theta(t) = \prod(1-te^\alpha)$ is W-symmetric, we have

$$Q(\mu;t)J(e^\rho)/\Theta(t) = J\Big(e^{\mu+\rho}\textstyle\prod_{\alpha>0}(1-te^{-\alpha})\Big).$$

We would now be done if we had used (1) as the definition of $P(\lambda;t)$; instead, our task is, in effect, to prove (1). Expanding the right hand side as a double sum over W_μ and coset representatives for W/W_μ, we see that

$$Q(\mu;t)J(e^\rho)/\Theta(t) = \sum_{x\in W^\mu} \operatorname{sgn}(x)x\Big(e^\mu \sum_{w\in W_\mu} \operatorname{sgn}(w)\prod_{\alpha>0}(e^{w\alpha/2} - te^{-w\alpha/2})\Big). \qquad (3)$$

Now consider $\Delta(t) := \sum_{w\in W} \operatorname{sgn}(w)\prod_{\alpha>0}(e^{w\alpha/2} - te^{-w\alpha/2})$. It is clear that $\gamma \leqslant \rho$ for every term $\pm t^l e^\gamma$ in the expansion of $\Delta(t)$, and e^ρ is the lowest possible dominant term in a skew-symmetric object, so $\Delta(t)$ must be a multiple of $J(e^\rho)$. Furthermore, $\gamma = \rho$ if and only if each time we select $w\alpha$ or $-w\alpha$, we choose the positive root. It is well-known that $\ell(w) = |\{\alpha > 0 : w\alpha < 0\}|$ and $\operatorname{sgn}(w) = (-1)^{\ell(w)}$, so we conclude that

$$\sum_{w\in W} \operatorname{sgn}(w)\prod_{\alpha>0}(e^{w\alpha/2} - te^{-w\alpha/2}) = \sum_{w\in W} \operatorname{sgn}(w)(-t)^{\ell(w)}J(e^\rho) = W(t)J(e^\rho). \qquad (4)$$

One may recognize the inner sum in (3) as the W_μ-version of (4), except that (3) has extra factors corresponding to the roots $\alpha \in \Phi^+$ whose reflections are not in W_μ; i.e., $\langle\mu,\alpha\rangle > 0$. However, these extra roots are permuted by W_μ, so the extra factors may be pulled out of the inner sum, yielding

$$\begin{aligned} Q(\mu;t)J(e^\rho)/\Theta(t) &= W_\mu(t)\sum_{x\in W^\mu} \operatorname{sgn}(x)x\Big(e^\mu \prod_{\langle\mu,\alpha\rangle>0}(e^{\alpha/2} - te^{-\alpha/2})J_\mu(e^{\rho_\mu})\Big) \\ &= W_\mu(t)\sum_{w\in W} \operatorname{sgn}(w)w\Big(e^{\mu+\rho}\prod_{\langle\mu,\alpha\rangle>0}(1-te^{-\alpha})\Big) = W_\mu(t)P(\mu;t), \end{aligned}$$

where J_μ and ρ_μ denote the W_μ-versions of J and ρ. $\square$

Let us introduce the standard symmetric pairing

$$(f,g) := \frac{1}{|W|}[e^0]\sum_{w\in W} fg^* J(e^\rho)J(e^\rho)^* \qquad (f,g\in\mathbb{Z}[\Lambda]),$$

where * is the ring involution defined by $(e^\mu)^* = e^{-\mu}$. It is well-known and an easy exercise to show that the Weyl characters $\chi(\lambda)$ are orthonormal with respect to this pairing.

Now extend $(\cdot,\cdot)$ linearly to a $\mathbb{Q}[[t]]$-valued pairing on $\mathbb{Z}[\Lambda][[t]]$ in the obvious way.

(vii) $(P(\lambda;t), Q(\mu;t)) = \delta_{\lambda,\mu}$ *for all* $\lambda, \mu \in \Lambda^+$.

Proof. Half of this is easy. From (ii) and (v), we know that $P(\lambda;t) = \chi(\lambda) + \text{lower terms}$ and $Q(\mu;t) = \chi(\mu) + \text{higher terms}$, so $(P(\lambda;t), Q(\mu;t))$ is 1 if $\lambda = \mu$ and 0 unless $\lambda \geqslant \mu$. However, $\Theta(t)^* = \Theta(t)$, so (vi) implies

$$\begin{aligned}(P(\lambda;t), Q(\mu;t)) &= W_\mu(t) \cdot (P(\lambda;t), \Theta(t)P(\mu;t)) = W_\mu(t) \cdot (\Theta(t)P(\lambda;t), P(\mu;t)) \\ &= W_\mu(t)/W_\lambda(t) \cdot (Q(\lambda;t), P(\mu;t)),\end{aligned}$$

so $(P(\lambda;t), Q(\mu;t))$ must vanish when $\lambda \neq \mu$ by symmetry. $\square$

As an immediate corollary, we obtain

(viii) $Q(\mu;t) = \sum_{\lambda \geqslant \mu} K_{\lambda,\mu}(t)\chi(\lambda)$.

Finally, to prove Kato's formula, note that (viii) implies that $K_{\lambda,\mu}(t)$ is the coefficient of $J(e^{\lambda+\rho})$ in $J(e^{\mu+\rho}\prod(1-te^\alpha)^{-1})$. Equivalently,

$$\begin{aligned}K_{\lambda,\mu}(t) &= [e^{\lambda+\rho}] \sum_{w \in W} \operatorname{sgn}(w) w\Big(e^{\mu+\rho} \prod_{\alpha>0}(1-te^\alpha)^{-1}\Big) \\ &= \sum_{w \in W} \operatorname{sgn}(w) w[e^{(\lambda+\rho)-w(\mu+\rho)}] \prod_{\alpha>0}(1-te^{w\alpha})^{-1} \\ &= \sum_{w \in W} \operatorname{sgn}(w) \mathcal{P}(w^{-1}(\lambda+\rho)-(\mu+\rho);t),\end{aligned}$$

thus proving (2).

3. Generalized Exponents

If G is a (complex, reductive) Lie group with Lie algebra $\mathfrak{g}$, then it is a classical theorem of Chevalley that the G-invariants in the symmetric algebra of the adjoint representation of G; i.e., $S(\mathfrak{g})^G$, is a polynomial ring with homogeneous generators. Moreover, the Poincaré series of W and the degrees of the generators, say $d_1, \ldots, d_n$, are related via

$$W(t) = \prod_{i=1}^{n} \frac{1-t^{d_i}}{1-t}.$$

Thus, $(1-t)^{-n}W(t)^{-1}$ is the Hilbert series of $S(\mathfrak{g})^G$.

For example, if $G = GL(n)$, then the adjoint representation of G is the conjugation action of G on $\mathfrak{g} = gl(n)$, and $S(\mathfrak{g})^G$ is the ring of polynomials in n^2 matrix entries that are invariant under conjugation. As free generators, one may take the coefficients of the characteristic polynomial of an $n \times n$ matrix.

Kostant proved that $S(\mathfrak{g})$ is free as a module over $S(\mathfrak{g})^G$. Since

$$\frac{1}{(1-t)^n}\Theta(t) = \frac{1}{(1-t)^n}\prod_{\alpha\in\Phi}\frac{1}{1-te^\alpha}$$

is the graded character of $S(\mathfrak{g})$ (recall $\Theta(t)$ from (vi)), it follows via (vi) that

$$Q(0;t) = W(t)\Theta(t)$$

is the graded character of $S(\mathfrak{g})/S(\mathfrak{g})^G_+$, where $S(\mathfrak{g})^G_+$ denotes the ideal of $S(\mathfrak{g})$ generated by invariants of positive degree. Thus via (viii) we obtain that $K_{\lambda,0}(t)$ is the graded multiplicity of the irreducible representation of G of highest weight λ in $S(\mathfrak{g})/S(\mathfrak{g})^G_+$. In particular, $K_{\lambda,0}(t)$ has nonnegative coefficients.

Kostant refers to $K_{\lambda,0}(t)$ as the generating series for "generalized exponents," since

$$K_{\widetilde{\alpha},0}(t) = t^{d_1-1} + \cdots + t^{d_n-1}$$

is the generating series for the usual exponents, where $\widetilde{\alpha}$ denotes the highest root.

There is an analogous but more complicated way to interpret $K_{\lambda,\mu}(t)$ for arbitrary (dominant) μ as a Hilbert series arising from a filtration of weight spaces in the irreducible G-representation of highest weight λ. For details, see [**B2**] and [**JLZ**].

4. Kazhdan-Lusztig Polynomials

A. The Big Affine Hecke Algebra.

Let $\widehat{W}$ denote the extended affine Weyl group generated by W and $T(\Lambda) = \{t_\lambda : \lambda \in \Lambda\}$, the latter acting as a group of (affine) translations on $\operatorname{Span}\Lambda$. It is an annoying fact of life that $\widehat{W}$ need not be a Coxeter group; however, it does include the "unextended" affine Weyl group W_a as a subgroup. The latter is generated by reflections through the affine hyperplanes $\langle\,\cdot\,,\alpha^\vee\rangle = k$ ($\alpha \in \Phi$, $k \in \mathbb{Z}$), and is a Coxeter group relative to the set S_a of reflections through the walls of the fundamental alcove

$$A_0 = \{\mu \in \operatorname{Span}\Lambda : 0 \leqslant \langle \mu, \alpha^\vee\rangle \leqslant 1 \text{ for all } \alpha \in \Phi^+\}.$$

This alcove is a fundamental domain for the action of W_a, and $\widehat{W}$ is the semidirect product of W_a and $\Omega = \{w \in \widehat{W} : wA_0 = A_0\}$.

The length function on W_a may be extended geometrically to $\widehat{W}$ by defining $\ell(w)$ to be the number of reflecting hyperplanes that separate A_0 and wA_0. In particular, Ω is the set of elements of length 0. This allows us to define a Bruhat ordering of $\widehat{W}$ by taking the transitive closure of the relations

$$w < sw \quad \text{whenever} \quad \ell(w) < \ell(sw)$$

for all $w \in \widehat{W}$ and all (affine) reflections $s \in W_a$.

Now let $\widehat{H}$ denote the Iwahori-type Hecke algebra of $\widehat{W}$ over the ground ring $\mathbb{Z}[t^{\pm 1/2}]$. This algebra has a basis of the form $\{T_w : w \in \widehat{W}\}$, and may be defined by the relations

$$\begin{aligned} T_xT_y &= T_{xy} \quad \text{if } \ell(x) + \ell(y) = \ell(xy), \\ (T_s+1)(T_s-t) &= 0 \quad \text{if } s \in S_a. \end{aligned}$$

Note that these relations over-determine the algebra; one needs to prove that no dependence relation among the T_w's is a consequence of these relations.

B. The Satake Transform.

Enlarging the ground ring of $\widehat{H}$ to $\mathbb{Q}(t^{1/2})$, define

$$\phi_\lambda := \frac{1}{W_\lambda(t)} \cdot \sum_{w \in W t_\lambda W} T_w \qquad (\lambda \in \Lambda).$$

In case $\lambda = 0$, this is a sum over W.

(ix) $\phi_0^2 = \phi_0$.

Proof. Show that $T_w \phi_0 = t^{\ell(w)} \phi_0$ for $w \in W$ (exercise). $\square$

(x) $\phi_0 \widehat{H} \phi_0 = \{\sum a_x T_x : a_x$ *depends only on* $W x W\}$.

Proof. Since (ix) implies $\phi_0 \widehat{H} \phi_0 = \phi_0 \widehat{H} \cap \widehat{H} \phi_0$, it suffices to prove the analogous one-sided version; i.e., a description of the ideal $\phi_0 \widehat{H}$. However, if y is the shortest element of the coset Wy, then $\ell(wy) = \ell(w) + \ell(y)$ for all $w \in W$, whence $W(t)\phi_0 T_y = \sum_{w \in W} T_{wy}$, and if we replace y with wy, we get a multiple of this sum. Therefore, $\sum a_x T_x \in \phi_0 \widehat{H}$ if and only if a_x depends only on Wx. $\square$

(xi) $\{\phi_\lambda : \lambda \in \Lambda^+\}$ *is a basis of the algebra* $\phi_0 \widehat{H} \phi_0$.

Proof. Certainly (x) implies $\phi_\lambda \in \phi_0 \widehat{H} \phi_0$. To see that we have a basis, note that $\widehat{W}$ is the semidirect product $T(\Lambda) \rtimes W$, so $T(\Lambda)$ is a set of coset representatives for $\widehat{W}/W$. Furthermore, the calculation $w t_\lambda w^{-1} = t_{w\lambda}$ ($w \in W$, $\lambda \in \Lambda$) implies that we obtain double coset representatives for $W \backslash \widehat{W} / W$ by taking one translation from each W-orbit. In particular, $\{t_\lambda : \lambda \in \Lambda^+\}$ will suffice. Now use (x). $\square$

Let $^- : \widehat{H} \to \widehat{H}$ denote the ring involution defined by $(t^{1/2})^- = t^{-1/2}$ and $\bar{T}_w = T_{w^{-1}}^{-1}$. Although it is not immediately obvious, the ring $\phi_0 \widehat{H} \phi_0$ is stable under this involution, and this turns out to be significant. (The proof will be given later.)

Fact. *The Satake Transform is an isomorphism between two of the following rings:*

(a) *The Hecke algebra $H(G, K)$ of K-bi-invariant functions on G, where G is a p-adic Chevalley group with root system $\Phi^\vee$ and maximal compact subgroup K.*
(b) $\phi_0 \widehat{H} \phi_0$.
(c) $Z(\widehat{H})$*; i.e., the center of* $\widehat{H}$.
(d) $\mathbb{Q}(t^{1/2})[\Lambda]^W$.

We have cheated a bit here, since the 't' in (a) is the scalar $1/p$, not an indeterminate.

It is irritating that there is a multitude of opinions in the literature about which pairs among these four rings the Satake Transform operates between, and the direction it flows.

If you are Satake, it is a map from (a) to (d) defined by an integral. If you are Lusztig (see [**L**]), it is an easy map from (c) to (b): $z \mapsto \phi_0 z = \phi_0 z \phi_0$ (but proving that it is an isomorphism is not completely trivial). In any case, everyone seems to agree that the Satake Transform is not a map between (a) and (b). An isomorphism between these two rings is relatively straightforward (in fact, ϕ_λ corresponds to a function supported only on the λ-th (K, K)-double coset in G). Also, the equivalence of (c) and (d) is a relatively easy calculation due to Bernstein. This leaves us with potentially eight different maps ((a) or (b) to or from (c) or (d)) that various authors might define as the "Satake Transform." I hope not to discover if all eight have advocates.

For us, the ring structure is immaterial. All we will need to know is that as a linear map from $\phi_0 \widehat{H} \phi_0$ to $\mathbb{Q}(t^{1/2})[\Lambda]^W$, the Satake Transform has the following two features:

$$\phi_\lambda \mapsto t^{\langle \lambda, \rho^\vee \rangle} P(\lambda; t^{-1}) \qquad (\lambda \in \Lambda^+), \tag{5}$$

$$\bar{\phi}_\lambda \mapsto t^{-\langle \lambda, \rho^\vee \rangle} P(\lambda; t) \qquad (\lambda \in \Lambda^+). \tag{6}$$

This is essentially the result of Macdonald we mentioned at the end of §1 (see also [**K**]).

C. The Kazhdan-Lusztig Basis of $\widehat{H}$.

The following is a slight variation on the original definition/theorem in [**KL**] (see [**L**]), since $\widehat{W}$ need not be a Coxeter group.

Definition/Theorem. *For each $w \in \widehat{W}$, there is a unique $C'_w \in \widehat{H}$ such that*

(a) $\bar{C}'_w = C'_w$,
(b) $t^{\ell(w)/2} C'_w = \sum_{x \leqslant w} P_{x,w}(t) T_w$, *where* $P_{x,w}(t) \in \mathbb{Z}[t]$,
(c) $P_{w,w}(t) = 1$, *and* $\deg P_{x,w}(t) \leqslant (\ell(w) - \ell(x) - 1)/2$ *for* $x < w$.

The transition matrix between C'_w and T_w is triangular, so it is clear that the C'_w's form a basis of $\widehat{H}$, given that they exist. The coefficients $P_{x,w}(t)$ are the Kazhdan-Lusztig polynomials for $\widehat{W}$; their coefficients are known to be ranks of intersection homology groups associated to affine-type flag varieties (and hence, nonnegative).

Our goal in this section is to prove that the Kostka-Foulkes polynomials are Kazhdan-Lusztig polynomials. More precisely,

$$K_{\lambda,\mu}(t) = t^{\langle \lambda - \mu, \rho^\vee \rangle} P_{w_\mu, w_\lambda}(t^{-1}) \qquad (\lambda, \mu \in \Lambda^+), \tag{7}$$

where w_λ denotes the longest element of $W t_\lambda W$. An equivalent formula was conjectured (and proved for $t = 1$) by Lusztig in [**L**], and proved in full generality by Kato [**K**].

Note that w_0 (i.e., w_λ in the case $\lambda = 0$) is the longest element of W, as it should be.

Given what has already been established, the proof needs three more ingredients.

(xii) $w \leqslant w_\lambda$ *if and only if* $w \in W t_\mu W$ *for some* $\mu \in \Lambda^+$ *such that* $\mu \leqslant \lambda$.

Proof. It is not difficult to prove this from known properties of the Bruhat order and the partial order of dominant weights; e.g., it is a corollary of Theorem 4.10 of [**S**]. □

(xiii) $t^{\ell(w_\lambda)/2}W(t)^{-1}C'_{w_\lambda} = \sum_{\mu\leqslant\lambda} P_{w_\mu,w_\lambda}(t)\phi_\mu$.

Proof. A basic property of the Kazhdan-Lusztig polynomials (see (2.3g) of [**KL**]) is that if w is the longest element in some left or right parabolic coset, then $P_{x,w}(t)$ depends as a function of x only on the corresponding left or right coset of x. In our context, this means that $P_{x,w_\lambda}(t)$ depends only on WxW, and hence (xii) and part (b) of the definition of C'_{w_λ} yields the claimed expansion. $\square$

Note that (xiii) shows that $\{C'_{w_\lambda} : \lambda\in\Lambda^+\}$ is another basis of $\phi_0\widehat{H}\phi_0$. Bearing in mind part (a) of the definition of C'_w, this confirms our previous remark that $\phi_0\widehat{H}\phi_0$ is stable under the bar involution. In particular, note that in the case $\lambda = 0$, we obtain

$$\phi_0 = \bar{\phi}_0 = t^{\ell(w_0)/2}W(t)^{-1}C'_{w_0},$$

since $W(t) = t^{\ell(w_0)}W(1/t)$.

(xiv) $\ell(w_\lambda) = \ell(w_0) + \langle\lambda, 2\rho^\vee\rangle$.

Proof. Note that $\{wA_0 : w\in Wt_\lambda W\}$ is the set of alcoves with a point on their boundary in the W-orbit of λ. By definition, $w_\lambda A_0$ is the alcove in this set that has the most hyperplanes separating it from A_0. Given that λ is dominant, it follows that $w_\lambda A_0$ must have $w_0\lambda$ as a vertex, and every point μ in this alcove must satisfy $\langle\mu,\alpha^\vee\rangle < \langle w_0\lambda,\alpha^\vee\rangle \leqslant 0$ for all $\alpha\in\Phi^+$. Hence there are $1+\langle\lambda,\alpha^\vee\rangle$ hyperplanes orthogonal to $\alpha>0$ that separate $w_\lambda A_0$ from A_0, and $\ell(w_\lambda) = \sum_{\alpha>0}(1+\langle\lambda,\alpha^\vee\rangle) = |\Phi^+| + \langle\lambda,2\rho^\vee\rangle$. $\square$

Proof of (7). Using (xiv) to rewrite (xiii), we have

$$\frac{t^{\ell(w_0)/2}}{W(t)}C'_{w_\lambda} = \sum_{\mu\leqslant\lambda} P_{w_\mu,w_\lambda}(t)t^{-\langle\lambda,\rho^\vee\rangle}\phi_\mu.$$

The left side is clearly bar-invariant, so

$$\sum_{\mu\leqslant\lambda} P_{w_\mu,w_\lambda}(t)t^{-\langle\lambda,\rho^\vee\rangle}\phi_\mu = \sum_{\mu\leqslant\lambda} P_{w_\mu,w_\lambda}(t^{-1})t^{\langle\lambda,\rho^\vee\rangle}\bar{\phi}_\mu.$$

Now apply the Satake Transform. More precisely, if we apply (5) and (6) to the two sides of this identity, we obtain

$$\sum_{\mu\leqslant\lambda} P_{w_\mu,w_\lambda}(t)t^{\langle\mu-\lambda,\rho^\vee\rangle}P(\mu;t^{-1}) = \sum_{\mu\leqslant\lambda} P_{w_\mu,w_\lambda}(t^{-1})t^{\langle\lambda-\mu,\rho^\vee\rangle}P(\mu;t). \tag{8}$$

The degree bounds on Kazhdan-Lusztig polynomials (part (c) of the definition) imply

$$\deg P_{w_\mu,w_\lambda}(t) < (\ell(w_\lambda)-\ell(w_\mu)-1)/2 < \langle\lambda-\mu,\rho^\vee\rangle \quad (\text{if } \mu<\lambda),$$

so $t^{\langle \lambda-\mu,\rho^\vee\rangle} P_{w_\mu,w_\lambda}(t^{-1})$ has only positive powers of t, unless $\lambda = \mu$. Thus in (8), the right side has only $\geqslant 0$ powers of t and the left side has only $\leqslant 0$ powers, so both expressions must equal their constant term; namely, $P(\lambda; 0) = \chi(\lambda)$ (see (iii)). Thus we obtain

$$\chi(\lambda) = \sum_{\mu \leqslant \lambda} P_{w_\mu,w_\lambda}(t^{-1}) t^{\langle \lambda-\mu,\rho^\vee\rangle} P(\mu; t),$$

and the result follows from the definition of $K_{\lambda,\mu}(t)$. $\square$

A corollary of the above proof is that $\chi(\lambda)$ is the image of $t^{\ell(w_0)/2} W(t)^{-1} C'_{w_\lambda}$ under the Satake Transform.

References

[**B1**] R. K. Gupta, Characters and the q-analog of weight multiplicity, *J. London Math. Soc.* **36** (1987), 68–76.

[**B2**] R. K. Brylinski, Limits of weight spaces, Lusztig's q-analogs, and fiberings of adjoint orbits, *J. Amer. Math. Soc.* **2** (1989), 517–533.

[**JLZ**] A. Joseph, G. Letzter and S. Zelikson, On the Brylinski-Kostant filtration, *J. Amer. Math. Soc.* **13** (2000), 945–970.

[**K**] S. Kato, Spherical functions and a q-analogue of Kostant's weight multiplicity formula, *Invent. Math.* **66** (1982), 461–468.

[**KL**] D. Kazhdan and G. Lusztig, Representations of Coxeter groups and Hecke algebras, *Invent. Math.* **53** (1979), 165–184.

[**L**] G. Lusztig, Singularities, character formulas, and a q-analog of weight multiplicities, *in* "Analysis and topology on singular spaces II–III," *Astérisque* **101–102** (1983), 208–229.

[**M1**] I. G. Macdonald, Spherical functions on a group of p-adic type, *Publications of the Ramanujan Institute, No. 2.* University of Madras, Madras, 1971.

[**M2**] I. G. Macdonald, "Symmetric Functions and Hall Polynomials," Second Edition, Oxford Univ. Press, Oxford, 1995.

[**EC1**] R. P. Stanley, "Enumerative Combinatorics, Vol. I," Cambridge Univ. Press, Cambridge, 1997.

[**S**] J. R. Stembridge, Tight quotients and double quotients in the Bruhat order, *Electron. J. Combin.* **11(2)** (2005), R14, 41pp.