Nilpotent Orbits and Commutative Elements

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Abstract. Let W be a simply-laced Coxeter group with generating set S, and let W_{c} denote the subset consisting of those elements whose reduced expressions have no substrings of the form sts for any non-commuting $s, t \in S$. We give a root system characterization of W_{c}, and in the case where W corresponds to a finite Weyl group, show that W_{c} is a union of Spaltenstein-Springer-Steinberg cells. The latter is valid also for affine Weyl groups of type A, but not for types D or E.

1. Introduction

Let W be a Coxeter group with (finite) generating set $S = \{s_i\}_{i\in I}$. In the Weyl group case, the "commutative" elements of W were defined in [**F1**] to be those elements having no reduced expression containing a substring of the form $s_i s_j s_i$, where s_i and s_j are (non-commuting) generators such that the simple root corresponding to s_j is at least as long as the simple root corresponding to s_i. The "fully commutative" elements of a general Coxeter group were defined in [**S1**] to be those elements having no reduced expression containing a substring $s_i s_j s_i s_j \cdots$ of length $m \geq 3$, where m is the order of $s_i s_j$ in W. In the simply-laced case these two definitions agree, since the product of any pair of generators has order 2 or 3, and all roots have the same length.

There are numerous characterizations and properties of (fully) commutative elements in [**F1**], [**F3**], [**S1**] and [**S2**]. In this paper, we extend some previous characterizations in [**F1**] for finite, simply-laced Coxeter groups to arbitrary simply-laced Coxeter groups. In particular, in Section 2, we provide a root system characterization of commutativity. (The special case corresponding to finite Weyl groups was first proved in [**F1**], by a different argument.) This can be viewed as a generalization of the fact that in the symmetric group, the commutative elements are the permutations with no decreasing subsequence of length 3.

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In Section 3, we study the relationship between commutative elements and certain nilpotent orbits in the associated Lie algebra when W is a (simply-laced) finite or affine Weyl group. In particular, we obtain that $W_{\rm c}$ is a union of Spaltenstein-Springer-Steinberg cells if and only if W is affine of type A, or finite.

2. Root System Characterizations of $W_{\rm c}$.

We assume henceforth that the Coxeter group W is simply-laced; thus $s_i s_j = s_j s_i$ or $s_i s_j s_i = s_j s_i s_j$ for all $i, j \in I$. Let Γ denote the Coxeter graph corresponding to W; i.e., the simple graph with vertex set I and i adjacent to j if and only if s_i and s_j do not commute. We let $W_{\rm c}$ denote the subset of W consisting of those elements with no reduced expression containing a substring $s_i s_j s_i$ for any adjacent pair i, j of Γ.

Let V be a vector space over $\mathbb{Q}$ with basis $\Pi = \{\alpha_i\}_{i\in I}$, and let $\langle\ ,\ \rangle$ denote the symmetric bilinear form on V defined by

$$\langle \alpha_i, \alpha_j\rangle = \begin{cases} \ \ 2 & \text{if } i = j, \\ -1 & \text{if } i \text{ and } j \text{ are adjacent in } \Gamma, \\ \ \ 0 & \text{otherwise.} \end{cases}$$

The space V carries the reflection representation of W; namely,

$$s_i\beta = \beta - \langle \beta, \alpha_i\rangle\alpha_i$$

for all $\beta \in V$, $i \in I$. Furthermore, $\langle\ ,\ \rangle$ is W-invariant relative to this action.

Let Φ denote the (generalized) root system generated by the action of W on Π; i.e., $\Phi = \{w\alpha_i \mid w \in W, i \in I\}$. Every $\alpha \in \Phi$ is an integer linear combination of the simple roots $\alpha_i \in \Pi$. Let Φ^+ denote the set of positive roots; i.e., the set of $\alpha \in \Phi$ whose coefficients relative to Π are nonnegative. For every root α, we have either $\alpha \in \Phi^+$ or $-\alpha \in \Phi^+$ (e.g., [**H**, §5.4]). We write $\alpha > 0$ and $\alpha < 0$ in these cases, respectively.

For $w \in W$, let $\Phi(w)$ denote the set of roots $\alpha > 0$ such that $w\alpha < 0$. The cardinality of $\Phi(w)$ is the length l of any reduced expression $w = s_{i_1}\cdots s_{i_l}$, also denoted $\ell(w)$. In fact $\Phi(w) = \{\gamma_1, \ldots, \gamma_l\}$, where

$$\gamma_1 = \alpha_{i_l},\ \gamma_2 = s_{i_l}\alpha_{i_{l-1}},\ \ldots\ ,\ \gamma_l = s_{i_l}\cdots s_{i_2}\alpha_{i_1}.$$

We refer to $(\gamma_1, \ldots, \gamma_l)$ as the *root sequence* of the reduced expression $s_{i_1}\cdots s_{i_l}$.

We remark that $\Phi(w)$ is "biconvex" (cf. [**Bj**, §3]) in the sense that for all $\alpha, \beta \in \Phi^+$ and all integers $c_1, c_2 > 0$ such that $c_1\alpha + c_2\beta \in \Phi^+$, we have

$$\begin{aligned} &\alpha, \beta \in \Phi(w) \Rightarrow c_1\alpha + c_2\beta \in \Phi(w) \\ &\alpha, \beta \notin \Phi(w) \Rightarrow c_1\alpha + c_2\beta \notin \Phi(w). \end{aligned} \tag{2.1}$$

In fact, these convexity properties characterize the finite subsets of Φ^+ of the form $\Phi(w)$ for some $w \in W$.

Lemma 2.1. *We have* $\langle \alpha, \beta \rangle \geq -1$ *for all* $\alpha, \beta \in \Phi(w)$.

Proof. If $\alpha, \beta \in \Phi(w)$ are roots such that $\langle \alpha, \beta \rangle = -c \leq -2$, then the reflection corresponding to α maps β to $\beta + c\alpha$, a root in the positive linear span of α and β. Hence $\beta + c\alpha \in \Phi(w)$, by (2.1). However $\langle \beta, \beta + c\alpha \rangle = 2 - c^2 \leq -2$, so iterations of the map $(\alpha, \beta) \mapsto (\beta, \beta + c\alpha)$ generate an infinite sequence in the finite set $\Phi(w)$. $\square$

Given a root sequence $(\gamma_1, \ldots, \gamma_l)$ for w, let us partially order $\Phi(w)$ by taking the transitive closure of the relations $\gamma_i < \gamma_j$ for all $i < j$ such that $\langle \gamma_i, \gamma_j \rangle \neq 0$.

Proposition 2.2. *The partial ordering of* $\Phi(w)$ *is independent of the choice of root sequence if and only if* $w \in W_{\mathrm{c}}$.

Proof. Any reduced expression for $w \in W$ can be obtained from any other by a sequence of braid moves (i.e., $s_i s_j s_i \to s_j s_i s_j$ or $s_i s_j \to s_j s_i$, according to whether i and j are adjacent in Γ) [**B**, §IV.1.5]. Therefore, if there are no opportunities to apply braid moves of length three (i.e., $w \in W_{\mathrm{c}}$), all reduced expressions for w can be generated merely by interchanging consecutive pairs of commuting generators. In the root sequence, these moves correspond to interchanging consecutive pairs of orthogonal roots and clearly have no effect on the partial order.

On the other hand, if i and j are adjacent in Γ, then the root sequences corresponding to the two reduced expressions for $x = s_i s_j s_i = s_j s_i s_j$ are $(\alpha_i, \alpha_i + \alpha_j, \alpha_j)$ and $(\alpha_j, \alpha_i + \alpha_j, \alpha_i)$, and the partial orders are total. It follows that if $s_i s_j s_i$ is a substring of some reduced expression for w (i.e., $w \notin W_{\mathrm{c}}$), then there exist root sequences for w containing W-conjugates of these two subsequences, and hence the corresponding partial orders differ. $\square$

Remark 2.3. The partial ordering of a root sequence is isomorphic to the dual of the "heap" (see [**S1**, §1]) of the corresponding reduced expression. In particular, it follows that the extensions of the partial order to a total order are the root sequences that can be generated from the given root sequence by interchanging consecutive pairs of orthogonal roots.

In the following, let $\prec$ denote the customary partial ordering of Φ in which $\alpha \prec \beta$ whenever $\beta - \alpha$ has nonnegative coordinates relative to the simple roots.

Theorem 2.4. *For* $w \in W$, *the following are equivalent.*

(a) $w \in W_{\mathrm{c}}$.
(b) $\langle \alpha, \beta \rangle \geq 0$ *for all* $\alpha, \beta \in \Phi(w)$.
(c) *There does not exist a triple* $\alpha, \beta, \alpha + \beta \in \Phi(w)$.

(d) *The partial ordering of $\Phi(w)$ relative to some (equivalently, every) root sequence is consistent with $\prec$ (i.e., $\alpha < \beta$ in $\Phi(w)$ implies $\alpha \prec \beta$).*

Proof. We demonstrate that the negations of these properties are equivalent.

$\neg$(a) $\Rightarrow$ $\neg$(b). If (a) fails, then w has a reduced expression of the form $xs_is_js_iy$ for some adjacent pair i, j. It follows that the corresponding root sequence includes $\alpha = y^{-1}\alpha_i$ and $\beta = y^{-1}s_is_j\alpha_i = y^{-1}\alpha_j$, for which $\langle \alpha, \beta \rangle = \langle \alpha_i, \alpha_j \rangle = -1$.

$\neg$(b) $\Rightarrow$ $\neg$(c). If $\alpha, \beta \in \Phi(w)$ are roots such that $\langle \alpha, \beta \rangle < 0$, then $\langle \alpha, \beta \rangle = -1$ by Lemma 2.1. Therefore $\alpha + \beta$ is a root (being the reflection of β through α), and hence by (2.1) must belong to $\Phi(w)$.

$\neg$(c) $\Rightarrow$ $\neg$(d). Every initial segment of a root sequence is also a root sequence, and hence the subset of $\Phi(w)$ formed by such an initial segment must satisfy (2.1). It follows that a set of roots of the form $\alpha, \beta, \alpha + \beta \in \Phi(w)$ must occur in the order $(\alpha, \alpha + \beta, \beta)$ or $(\beta, \alpha + \beta, \alpha)$ in every root sequence, and hence also in the corresponding partial order. However, neither of these orderings is consistent with $\prec$.

$\neg$(d) $\Rightarrow$ $\neg$(a). If (d) fails, then there is a root sequence for w whose partial order includes a covering relation $\alpha < \beta$ that is not consistent with $\prec$; in particular, $\beta - \alpha \notin \Phi^+$. By choosing a suitable linear extension of the partial order, we may obtain a root sequence for w in which α and β appear consecutively, and hence $\alpha = y^{-1}\alpha_i$, $\beta = y^{-1}s_i\alpha_j$, given that the corresponding reduced expression for w is of the form xs_js_iy. Since $\langle \alpha, \beta \rangle \neq 0$ (otherwise $\alpha < \beta$ could not be a covering relation), it follows that

$$\langle \alpha, \beta \rangle = \langle y^{-1}\alpha_i, y^{-1}s_i\alpha_j \rangle = -\langle \alpha_i, \alpha_j \rangle = 1.$$

Hence $\alpha - \beta = -y^{-1}\alpha_j$ is a root, necessarily positive, since $\beta - \alpha \notin \Phi^+$. However, $y^{-1}\alpha_j < 0$ implies that there is a reduced expression for y that begins with s_j (e.g., [**H**, §5.4]). Hence there is a reduced expression for w containing the substring $s_js_is_j$, and $w \notin W_{\mathrm{c}}$. $\square$

Remark 2.5. The previous result can be viewed as a generalization of the fact that the commutative elements of the symmetric group S_n are the permutations $w = (w_1, \ldots, w_n)$ of $\{1, \ldots, n\}$ that do not contain a decreasing subsequence of length 3. Indeed, using $\{\varepsilon_j - \varepsilon_i \mid 1 \leq i < j \leq n\}$ as the set of positive roots for A_{n-1}, one sees that the triples of positive roots of the form $\alpha, \beta, \alpha + \beta$ are $\varepsilon_j - \varepsilon_i, \varepsilon_k - \varepsilon_j, \varepsilon_k - \varepsilon_i$, where $1 \leq i < j < k \leq n$. Having such a triple occur in $\Phi(w)$ is equivalent to having $w_i > w_j > w_k$. A similar description can be provided in type D; see [**F1**, §7] or [**S2**, §10].

3. Cells

Now suppose that W is the Weyl group of a semisimple, simply-laced, simply connected algebraic group G over $\mathbb{C}$ with Lie algebra $\mathfrak{g}$. We may assume that Φ is the root system of $\mathfrak{g}$ relative to some choice of Cartan subalgebra $\mathfrak{h}$, and that $\mathfrak{b}$ is the Borel subalgebra corresponding to the chosen simple roots Π.

Let $\mathfrak{n}$ be the nilpotent radical of $\mathfrak{b}$, and define $\mathfrak{n}_0 = w_0 \mathfrak{n} w_0$, where w_0 denotes the longest element of W. For $w \in W$, set $\mathfrak{n}_0^w = w \mathfrak{n}_0 w^{-1}$.

Let $\mathcal{N}$ be the subvariety of nilpotent elements in $\mathfrak{g}$, and let $\mathcal{N}/G$ denote the G-orbits of such elements. Following Spaltenstein, Springer, and Steinberg (et. al.), one may define a map $\phi : W \to \mathcal{N}/G$ by taking $\phi(w)$ to be the (unique) nilpotent orbit O such that $O \cap \mathfrak{n}_0^w \cap \mathfrak{n}$ is dense in $\mathfrak{n}_0^w \cap \mathfrak{n}$. (This differs from the standard definition by a factor of w_0.) The fibers of ϕ are cells.

We now pass to analogous structures for the affine Weyl group $\hat{W}$. It should be noted that $\hat{W}$ is also simply-laced (in the sense of §2) unless W is of type A_1. In this exceptional case, we can maintain the validity of Theorem 2.4 by defining $\hat{W}_{\mathrm{c}} := \hat{W}$.

Let $\hat{G} = G(F)$, where $F = \mathbb{C}((t))$. The abstract root system $\hat{\Phi}$ generated by $\hat{W}$ (in the sense of §2) can be identified with the real roots of the Lie algebra $\mathfrak{g} \otimes_{\mathbb{C}} F$.

Let $\hat{\mathfrak{b}}$ be the Iwahori subalgebra which sits in $\mathfrak{g} \otimes \mathbb{C}[[t]]$ as the inverse image of $\mathfrak{b}$ relative to the canonical projection $t \mapsto 0$, and let $\hat{\mathfrak{n}} \subset \hat{\mathfrak{b}}$ be the inverse image of $\mathfrak{n}$ relative to the same projection. Let $\hat{\mathfrak{n}}_0$ be the inverse image of $\mathfrak{n}_0$ relative to the canonical projection $\mathfrak{g} \otimes \mathbb{C}[t^{-1}] \to \mathfrak{g}$ defined by $t \mapsto \infty$. For $w \in \hat{W}$, we set $\hat{\mathfrak{n}}_0^w = w \hat{\mathfrak{n}}_0 w^{-1}$. For further details on this setup, see [**KL**, §0].

Each nilpotent orbit O in $\mathcal{N}/G$ also indexes a $G(\bar{F})$-orbit $\hat{O}$, where $\bar{F}$ denotes the algebraic closure of F. Following Lusztig [**L**], we may define a map $\hat{\phi} : \hat{W} \to \mathcal{N}/G$ by taking $\hat{\phi}(w)$ to be the (unique) nilpotent orbit O such that $\hat{O} \cap \hat{\mathfrak{n}}_0^w \cap \hat{\mathfrak{n}}$ is dense in $\hat{\mathfrak{n}}_0^w \cap \hat{\mathfrak{n}}$. Note that $\hat{\mathfrak{n}}_0^w \cap \hat{\mathfrak{n}}$ is finite-dimensional over $\mathbb{C}$; in fact, it is spanned by the root spaces indexed by $\hat{\Phi}(w^{-1})$.

Let $\mathcal{N}_4 = \{n \in \mathcal{N} \mid \mathrm{ad}(n)^4 = 0\}$, and let $\mathcal{N}_4/G$ denote the nilpotent orbits in $\mathcal{N}_4$.

Theorem 3.1. *We have*

(a) $W_{\mathrm{c}} = \phi^{-1}(\mathcal{N}_4/G)$.
(b) $\hat{W}_{\mathrm{c}} \supset \hat{\phi}^{-1}(\mathcal{N}_4/G)$, *with equality if and only if* W *is of type* A.

Let E_α denote a generator for the root space corresponding to $\alpha \in \hat{\Phi}$.

Lemma 3.2. *For* $w \notin \hat{W}_{\mathrm{c}}$, *there exists* $n \in \hat{\mathfrak{n}}_0^w \cap \hat{\mathfrak{n}}$ *such that* $\mathrm{ad}(n)^4 \neq 0$.

Proof. Given that $w \notin \hat{W}_{\mathrm{c}}$ (and hence $w^{-1} \notin \hat{W}_{\mathrm{c}}$), Theorem 2.4 implies that there is a triple $\alpha, \beta, \alpha+\beta \in \hat{\Phi}(w^{-1})$. This given, we take $n := E_\alpha + E_\beta \in \hat{\mathfrak{n}}_0^w \cap \hat{\mathfrak{n}}$. Since n is the regular element of an $\mathfrak{sl}_3$ subalgebra, it follows that $\mathrm{ad}(n)^4 \neq 0$ (e.g., see [**K**]).

Alternatively, one can directly compute $\mathrm{ad}(n)^4 E_{-\alpha-\beta}$ and verify that it is a non-zero multiple of $E_{\alpha+\beta}$. $\square$

LEMMA 3.3. *In an affine root system of type D or E, there exists a quadruple of orthogonal simple roots $\gamma_1, \ldots, \gamma_4$ and a root ρ such that $\langle \rho, \gamma_i \rangle = -1$ for all i.*

Proof. For type D, we may take $\gamma_1, \ldots, \gamma_4$ to be the simple roots corresponding to the four end nodes of Γ, and ρ to be the sum of the remaining simple roots.

For E_m, use $I = \{0, 1, \ldots, m\}$, with the indexing arranged so that 4 labels the node of degree three, 0 labels the node corresponding to the highest root, and $1, 3, 4, \ldots, m-4$ labels a path in Γ.

In E_6, it suffices to take $\{\gamma_1, \ldots, \gamma_4\} = \{\alpha_0, \alpha_1, \alpha_4, \alpha_6\}$ and $\rho = \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5$; in E_7, $\{\gamma_1, \ldots, \gamma_4\} = \{\alpha_0, \alpha_3, \alpha_5, \alpha_7\}$ and $\rho = \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6$; in E_8, $\{\gamma_1, \ldots, \gamma_4\} = \{\alpha_0, \alpha_2, \alpha_5, \alpha_7\}$ and $\rho = \alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7 + \alpha_8$. $\square$

Proof of Theorem 3.1. Lemma 3.2 implies that $\hat{\phi}^{-1}(\mathcal{N}_4/G) \subset \hat{W}_{\mathrm{c}}$, and essentially the same argument proves $\phi^{-1}(\mathcal{N}_4/G) \subset W$. To prove the reverse inclusions, it would suffice to show that for $w \in \hat{W}_{\mathrm{c}}$ and $n \in \hat{\mathfrak{n}}_0^w \cap \hat{\mathfrak{n}}$ (resp., $w \in W_{\mathrm{c}}$ and $n \in \mathfrak{n}_0^w \cap \mathfrak{n}$), we have $n \in \mathcal{N}_4$.

Since any $n \in \hat{\mathfrak{n}}_0^w \cap \hat{\mathfrak{n}}$ is a linear combination of those E_α such that $\alpha \in \hat{\Phi}(w^{-1})$, it follows that $\mathrm{ad}(n)^4$ is a linear combination of monomials of the form

$$M = \mathrm{ad}(E_{\gamma_1})\,\mathrm{ad}(E_{\gamma_2})\,\mathrm{ad}(E_{\gamma_3})\,\mathrm{ad}(E_{\gamma_4}), \tag{3.1}$$

where $\gamma_1, \ldots, \gamma_4 \in \hat{\Phi}(w^{-1})$.

If $\mathrm{ad}(n)^4 \neq 0$, at least one such monomial must be nonzero. Let us therefore suppose $M(E_\rho) \neq 0$ for some $\rho \in \hat{\Phi} \cup \{0\}$, following the convention that E_0 represents an arbitrary member of $\mathfrak{h} \otimes F$. Setting $\delta = \rho + \sum_i \gamma_i$, it is clearly necessary that $\delta \in \hat{\Phi} \cup \{0\}$. Furthermore,

$$\langle \delta, \delta \rangle = \langle \rho, \rho \rangle + 8 + 2\sum_i \langle \rho, \gamma_i \rangle + 2\sum_{i<j} \langle \gamma_i, \gamma_j \rangle \geq \langle \rho, \rho \rangle + 8 + 2\sum_i \langle \rho, \gamma_i \rangle, \tag{3.2}$$

since $\langle \gamma_i, \gamma_j \rangle \geq 0$ by Theorem 2.4.

If $\rho = 0$, this implies $\langle \delta, \delta \rangle \geq 8$, which is impossible. Hence $\rho \in \hat{\Phi}$ and $\langle \rho, \rho \rangle = 2$.

Since $\langle\ ,\ \rangle$ is positive semidefinite, it follows that $\langle \rho, \gamma_i \rangle \geq -2$ for all i. If $\langle \rho, \gamma_1 \rangle = -2$, then $\rho + \gamma_1$ would belong to the radical of $\langle\ ,\ \rangle$, and therefore

$$\langle \delta, \delta \rangle = \langle \gamma_2 + \gamma_3 + \gamma_4, \gamma_2 + \gamma_3 + \gamma_4 \rangle \geq 6,$$

a contradiction. Thus $\langle \rho, \gamma_i \rangle \geq -1$ and the bound implied by (3.2) yields $\langle \delta, \delta \rangle \geq 2$. This bound is tight, so equality occurs in (3.2); in particular, the γ_i must be pairwise orthogonal and $\langle \rho, \gamma_i \rangle = -1$ for all i. Conversely, in any such configuration of roots, we have $\langle \gamma_{i+1} + \cdots + \gamma_4 + \rho, \gamma_i \rangle = -1$, so $\gamma_i + \cdots + \gamma_4 + \rho \in \hat{\Phi}$ for all i, and hence $M(E_\rho) \neq 0$. Furthermore, if we set $n := E_{\gamma_1} + \cdots + E_{\gamma_4}$, then the above analysis shows that every term in the expansion of $\mathrm{ad}(n)^4$ is 0 except for the 24 monomials that correspond to selecting a permutation of (3.1). However, $\mathrm{ad}(E_{\gamma_i})$ and $\mathrm{ad}(E_{\gamma_j})$ commute pairwise for $i \neq j$, so $\mathrm{ad}(n)^4 = 24M \neq 0$.

If W is of type A, we claim that there can be no configuration $\rho, \gamma_1, \ldots, \gamma_4 \in \hat{\Phi}$ as above. Indeed, since the inner products among these roots coincide with those formed by the simple roots of an affine system of type D_4, they generate either a finite or affine subsystem of type D_4 in $\hat{\Phi}$, according to whether ρ is in the linear span of $\gamma_1, \ldots, \gamma_4$. In either case, modulo the radical of $\langle\ ,\ \rangle$, we would have an embedding of a finite root system of type D_4 in a finite root system of type A, which is impossible—every irreducible subsystem in type A is also of type A.

If W is of type D or E, Lemma 3.3 implies that there is a suitable configuration of roots $\rho, \gamma_1, \ldots, \gamma_4$ in which the γ_i are simple. If we take w to be the product of the simple reflections corresponding to the γ_i, it is clear that $w = w^{-1} \in \hat{W}_{\mathrm{c}}$ and $\hat{\Phi}(w) = \{\gamma_1, \ldots, \gamma_4\}$. Hence there exists $n \in \hat{\mathfrak{n}}_0^w \cap \hat{\mathfrak{n}}$ such that $\mathrm{ad}(n)^4 \neq 0$, and the inclusion in (b) is proper.

Turning now to (a), the above reasoning also proves that for $w \in W_{\mathrm{c}}$ and $n \in \mathfrak{n}_0^w \cap \mathfrak{n}$, we have $\mathrm{ad}(n)^4 = 0$ unless there exist pairwise orthogonal roots $\gamma_1, \ldots, \gamma_4 \in \Phi(w)$ and $\rho \in \Phi$ satisfying $\langle \rho, \gamma_i \rangle = -1$ for all i. However in this case $\langle\ ,\ \rangle$ is positive definite, so $\rho, \gamma_1, \ldots, \gamma_4$ must generate a *finite* root system Δ of type D_4.

Setting $\Delta^+ = \Phi^+ \cap \Delta$, we can choose an orthogonal basis $\varepsilon_1, \ldots, \varepsilon_4$ for the span of Δ so that $\Delta^+ = \{\varepsilon_i \pm \varepsilon_j \mid 1 \leq i < j \leq 4\}$. There are three quadruples of pairwise orthogonal roots in Δ^+; namely, $\{\varepsilon_i \pm \varepsilon_j, \varepsilon_k \pm \varepsilon_l\}$, where $\{\{i,j\},\{k,l\}\}$ ranges over the three partitions of $\{1, \ldots, 4\}$ into doubletons. We claim that if any of these configurations occurs in $\Phi(w)$, then there would exist a root $\beta \in \Phi(w)$ such that $\langle \beta, \gamma_i \rangle = -1$ for some i, contradicting the fact that $w \in W_{\mathrm{c}}$ (cf. Theorem 2.4).

If $\varepsilon_1 \pm \varepsilon_2, \varepsilon_3 \pm \varepsilon_4 \in \Phi(w)$, then the decomposition $\varepsilon_1 + \varepsilon_2 = (\varepsilon_1 - \varepsilon_4) + (\varepsilon_2 + \varepsilon_4)$ together with the convexity properties of (2.1) imply $\varepsilon_1 - \varepsilon_4$ or $\varepsilon_2 + \varepsilon_4 \in \Phi(w)$. However $\langle \varepsilon_1 - \varepsilon_4, \varepsilon_3 + \varepsilon_4 \rangle = -1$ and $\langle \varepsilon_2 + \varepsilon_4, \varepsilon_3 - \varepsilon_4 \rangle = -1$, so both cases lead to a contradiction.

Similarly, if $\varepsilon_1 \pm \varepsilon_3, \varepsilon_2 \pm \varepsilon_4 \in \Phi(w)$, then the decomposition $\varepsilon_1 + \varepsilon_3 = (\varepsilon_1 - \varepsilon_2) + (\varepsilon_2 + \varepsilon_3)$ and convexity together imply $\varepsilon_1 - \varepsilon_2$ or $\varepsilon_2 + \varepsilon_3 \in \Phi(w)$. However $\langle \varepsilon_1 - \varepsilon_2, \varepsilon_2 + \varepsilon_4 \rangle = -1$ and $\langle \varepsilon_2 + \varepsilon_3, \varepsilon_1 - \varepsilon_3 \rangle = -1$, so again both cases yield contradictions.

Finally, if $\varepsilon_1 \pm \varepsilon_4, \varepsilon_2 \pm \varepsilon_3 \in \Phi(w)$, then the decomposition $\varepsilon_2 + \varepsilon_3 = (\varepsilon_2 - \varepsilon_4) + (\varepsilon_3 + \varepsilon_4)$

and convexity imply $\varepsilon_2 - \varepsilon_4$ or $\varepsilon_3 + \varepsilon_4 \in \Phi(w)$. However $\langle \varepsilon_2 - \varepsilon_4, \varepsilon_1 + \varepsilon_4 \rangle = -1$ and $\langle \varepsilon_3 + \varepsilon_4, \varepsilon_1 - \varepsilon_4 \rangle = -1$, so both cases yield contradictions. $\square$

REMARK 3.4. (a) Part (a) of Theorem 3.1 was first proved in [**F1**, §7] and used there to determine the longest elements in W_c for finite simply-laced Weyl groups W.

(b) If W is of type D or E, Theorem 3.1(b) implies that there exists $w \in \hat{W}_c$ such that $\hat{\phi}(w) = O \notin \mathcal{N}_4/G$. On the other hand, it is known that every fiber of the map ϕ is non-empty, so $\phi^{-1}(O)$ must contain elements not in $W_{\rm c}$, by Theorem 3.1(a). Since ϕ and $\hat{\phi}$ commute with the natural embedding of W in $\hat{W}$, it follows that in these cases, $\hat{W}_{\rm c}$ is not a union of cells.

(c) From the tables in [**C**], it can be shown that for E_8 there are exactly five nilpotent orbits in $\mathcal{N}_4$. On the other hand, it is known that there are only finitely many commutative elements in the affine Weyl group $\hat{E}_8$ [**F1**, §3]. (In fact, there are exactly 44,199 such elements.) Thus $\hat{E}_8$ has at least five finite cells.

(d) The members of $\mathcal{N}_4$ are precisely the "spherical" nilpotents as classified by Panyushev [**P**]. (A nilpotent element is spherical if its orbit under the action of some Borel subgroup is dense in its G-orbit.) Panyushev's classification is achieved on a case-by-case basis; it is possible that further analysis of the fibers of ϕ will lead to a uniform proof, at least in the simply-laced cases.

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