Strange enumerations of CSPP's and TSPP's

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A Note to the Reader

This unfinished article is a report of some observations and conjectures I made several years ago concerning weighted enumerations of cyclically symmetric and totally symmetric plane partitions. The section concerned with the totally symmetric case is still unwritten and needs to be extracted from my hand-written notes. By popular demand, I am now making this article publically available.

John Stembridge, 27 September 1998

Preliminaries

Consider the elements of $\mathbf{P}^3$, regarded as the lattice points of $\mathbf{R}^3$ in the positive orthant. We can classify these points according to whether they lie on the positive side, negative side, or exactly meet each of the three hyperplanes $x = y$, $y = z$ and $x = z$. This partitions $\mathbf{P}^3$ into 13 families, corresponding to the thirteen faces of the Coxeter complex of type A_2. Given a subgroup G of S_3, acting on $\mathbf{P}^3$ in the obvious way, the number of distinct families of lattice points, modulo the action of G, shrinks to 8 for $G = S_2$, 5 for $G = C_3$, or 4 for $G = S_3$.

Let us fix a particular choice for G (possibly the trivial group), and let

$$\mathbf{P}^3/G = P_1 \cup \cdots \cup P_k$$

denote the partition of the G-orbits of $\mathbf{P}^3$ into the families described above, in some satisfactory order. Given a box $B = [a] \times [b] \times [c]$, let us define the generating function

$$N_B(x_1, \ldots, x_k) = \sum_{\pi \subset B} x_1^{m_1(\pi)} \ldots x_k^{m_k(\pi)},$$

where the sum ranges over all G-invariant plane partitions π that fit inside the box B, and $m_i(\pi) = |\pi/G \cap P_i|$ denotes the number of G-orbits of lattice points of π that belong to the ith family. If $B = [a]^3$, we will simply write $N_a(x_1, \dots, x_k)$.

All of the nice generating functions for G-invariant plane partitions are obtained by taking the x_i's to be certain powers of q.

CLAIM 1. *For each $G \subset S_3$, $N_B(x_1, \dots, x_k)$ can be expressed as a determinant or Pfaffian of size approximately a (or equivalently also b or c).*

CSPP

Now take $G = C_3$ and $B = [a]^3$. Orbit representatives for the five distinct types of lattice points are as follows:

$$(i, i, i), \quad (i, i, j), \quad (i, j, j), \quad (i, j, k), \quad (k, j, i),$$

where $i < j < k$. We will label these as types 1 through 5, respectively.

REMARK 2. Note that $N_a(x_1, \dots, x_5)$ is a symmetric function of x_4 and x_5. Although it is not a symmetric function of x_2 and x_3, it is easy to check that for any C_3-invariant plane partition $\pi \subset [a]^3$, we have

$$m_1(\pi) + m_1(\pi^c) = a,$$
$$m_2(\pi) + m_3(\pi^c) = \binom{a}{2},$$
$$m_4(\pi) + m_5(\pi^c) = \binom{a}{3},$$

where π^c denotes the complement of π. Consequently,

$$N_a(x_1, x_2, x_3, x_4, x_5) = x_1^a (x_2 x_3)^{\binom{a}{2}} (x_4 x_5)^{\binom{a}{3}} N_a(1/x_1, 1/x_3, 1/x_2, 1/x_5, 1/x_4).$$

Therefore, if N_a has a nice evaluation for some choice of $x_1, \dots, x_5$, then the same is true upon interchanging x_4 and x_5, or x_2 and x_3, and the results one obtains are essentially equivalent.

REMARK 3. Note that $N_a(q, q^3, q^3, q^3, q^3)$ is the usual generating function for C_3-invariant plane partitions, and thus is known to have a nice evaluation, thanks to Mills-Robbins-Rumsey. Also, the orbit-generating function is $N_a(q, q, q, q, q)$, which is known to not be nice.

Let $M(q) = [m_{ij}]_{0 \le i,j < a}$ denote the $a \times a$ matrix with

$$m_{ij}(q) = q^{\binom{j}{2}} \begin{bmatrix} i \\ j \end{bmatrix}_q,$$

and let $D(q) = \mathrm{diag}(1, q, q^2, \ldots, q^{a-1})$.

Lemma 4. *We have*

$$N_a(x_1, x_2, x_3, x_4, x_5) = \det[I + x_1 D(x_2) M(x_4) D(x_3) M(x_5)^t].$$

Partly motivated by the $q = -1$ phenomenon, we have investigated the behavior of $N_a(x_1 \ldots, x_5)$ for all possible specializations of the form $x_i = \pm 1$, looking for nice product formulas, as well as other numerological phenomena such as squarish numbers. By the above remark, there are really only 18 distinct special cases, not 32. The 18 cases arise from two choices for x_1, three choices for $\{x_2, x_3\}$, and three choices for $\{x_4, x_5\}$. We have found, empirically, a considerable amount of interesting behavior, which we briefly summarize before presenting the details, mostly in the form of data.

1. There are four cases for which there are, or appear to be, nice closed formulas. These four cases are characterized by the condition $x_2 = x_3 = x_4 = x_5$.

2. The six cases satisfying the condition $x_2 \neq x_3$ appear to be almost completely garbage. The factorizations are often ugly[1], and have no discernible relationship between neighboring terms (i.e., $\gcd(N_a, N_{a+1}), \gcd(N_a, N_{a+2}), \ldots$ tend to be very small). Also, each of these cases appears to have the property that N_a is never 0 for $a > 1$, and even the sign of N_a appears to obey no recognizable pattern. The only interesting phenomenon we have found that involves any of these six cases is the following.

Conjecture 5. $N_{2a}(-1, -1, 1, -1, -1) = (-1)^a N_{2a}(1, -1, 1, -1, -1)$.

3. The eight remaining cases can be further subdivided into two families: one of the families involves "squarish" numbers, and the other family exhibits a phenomenon that is much stranger and harder to describe. In this strange family, one finds neighboring terms (usually consecutive) that periodically and regularly have extremely large gcd's. These gcd's sometimes involve prime divisors that are so large, relative to N_a, one is forced to believe that there must be some deeper phenomenon that explains this. It is almost true that the strange family is characterized by the conditions $x_2 = x_3$ and $x_4 \neq x_5$; it does include all of these cases, but it also includes the exceptional case $(1, -1, -1, 1, 1)$. The remaining three cases form the squarish family. Also, most of these cases seem to have the property that the zeros of N_a occur periodically (and predictably), with a period of either 2 or 4.

We have also studied numerous specializations of $N_a(x_1, \ldots, x_5)$ in which $x_i = \pm q^{a_i}$ for various choices of a_i, but have found no examples in which nice factorizations occur (for all

[1] A technical definition: a positive integer n is *ugly* if it is the product of one large prime and a small number of small primes.

small a), other than cases that are corollaries of the Mills-Robbins-Rumsey factorization of $N_a(q, q^3, q^3, q^3, q^3)$.

In the following tables, we list, for each of the 18 cases, the values of N_a for $a = 1, \ldots, 20$, along with the prime factorizations of these numbers, as well as a brief indication of which of the above families the case belongs to, and any conjectures we have or theorems we know about them. We have not yet attempted to *prove* any of the conjectures; some of them could turn out to be easy consequences of Lemma 4.

Case 1: $[-1,-1,-1,-1,-1]$. Nice Formula.

1	0	0
2	1	1
3	0	0
4	4	2^2
5	0	0
6	49	7^2
7	0	0
8	1764	$2^2 3^2 7^2$
9	0	0
10	184041	$3^2 11^2 13^2$
11	0	0
12	55294096	$2^4 11^2 13^4$
13	0	0
14	47675849104	$2^4 13^4 17^2 19^2$
15	0	0
16	117727187246656	$2^6 13^2 17^4 19^4$
17	0	0
18	831443906113411600	$2^4 5^2 17^4 19^6 23^2$
19	0	0
20	16779127803917965290000	$2^4 3^2 5^4 7^2 17^2 19^6 23^4$

Theorem. *$N_{2a+1}=0$, $N_{2a}=D_a^2$, where*

$$D_a = \prod_{1\le i\le j\le a} \frac{a+i+j-1}{2i+j-1}.$$

Proof. This is can be deduced from known results (cf. Remark 3). Note that it is a special case of the $q=-1$ phenomenon. □

The numbers D_a are our old friends 1,2,7,42,429,7436....

Case 9: $[-1,1,1,1,1]$. Nice Formula.

1	0	0
2	-1	-1
3	0	0
4	16	2^4
5	0	0
6	-2401	-7^4
7	0	0
8	3111696	$2^4 3^4 7^4$
9	0	0
10	-33871089681	$-3^4 11^4 13^4$
11	0	0
12	3057437052457216	$2^8 11^4 13^8$
13	0	0
14	-2272986587787377602816	$-2^8 13^8 17^4 19^4$
15	0	0
16	13859690617009203134183182336	$2^{12} 13^4 17^8 19^8$
17	0	0
18	$-691298969013127603392281391014560000$	$-2^8 5^4 17^8 19^{12} 23^4$
19	0	0
20	281539129860212920649098576013644784100000000	$2^8 3^4 5^8 7^4 17^4 19^{12} 23^8$

Conjecture. *$N_{2a+1} = 0$, $N_{2a} = (-1)^a D_a^4$, where D_a is as in the previous case.*

Greg Kuperberg [private communication] claims to have a proof of this.

Case 10: $[1,-1,-1,-1,-1]$. Nice Formula.

1	2	2
2	3	3
3	8	2^3
4	16	2^4
5	72	$2^3 3^2$
6	243	3^5
7	1800	$2^3 3^2 5^2$
8	10000	$2^4 5^4$
9	125000	$2^3 5^6$
10	1171875	3.5^8
11	24500000	$2^5 5^6 7^2$
12	384160000	$2^8 5^4 7^4$
13	13553164800	$2^9 3^2 5^2 7^6$
14	358616740608	$2^8 3^5 7^8$
15	21253530912768	$2^{11} 3^6 7^6 11^2$
16	944697761796096	$2^{12} 3^8 7^4 11^4$
17	94479415952689152	$2^{11} 3^{12} 7^2 11^6$
18	7086679253206426368	$2^8 3^{17} 11^8$
19	1192720197932660187648	$2^9 3^{12} 11^{10} 13^2$
20	150555156342746717143296	$2^8 3^8 11^{12} 13^4$

Conjecture. *$N_{4a} = B_a^4$, $N_{4a+1} = 2B_a^2 C_a^2$, $N_{4a+2} = 3C_a^4$, $N_{4a+3} = 2B_{a+1}^2 C_a^2$, where*

$$B_a = \prod_{j=0}^{a-1} \frac{(3j+2)!(3j)!(j!)^2}{(2j+1)!^2(2j)!^2},$$
$$C_a = \prod_{j=1}^{a} \frac{(3j)!(3j-1)!j!(j-1)!}{(2j)!^2(2j-1)!^2}.$$

Case 18: $[1,1,1,1,1]$. Nice Formula.

1	2	2
2	5	5
3	20	$2^2 5$
4	132	$2^2 3.11$
5	1452	$2^2 3.11^2$
6	26741	$11^2 13.17$
7	826540	$2^2 5.11.13.17^2$
8	42939620	$2^2 5.17^3 19.23$
9	3752922788	$2^2 17^3 19^2 23^2$
10	552176360205	$3.5.17^2 19^2 23^3 29$
11	136830327773400	$2^3 3^2 5^2 17.19.23^4 29^2$
12	57125602787130000	$2^4 3^3 5^4 23^4 29^3 31$
13	40191587143536420000	$2^5 3^5 5^4 23^3 29^4 31^2$
14	47663133295107416936400	$2^4 3^5 5^2 23^2 29^5 31^3 37.41$
15	95288872904963020131203520	$2^6 3^3 5.11.23.29^5 31^4 37^2 41^2$
16	321195665986577042490185260608	$2^6 3^2 11^2 29^4 31^4 37^3 41^3 43.47$
17	1825621025091970952461683189714240	$2^6 3.5.11^2 29^3 31^3 37^4 41^4 43^2 47^2$
18	17498580094680296849166491514620858960	$2^4 5.7.11.29^2 31^2 37^5 41^5 43^3 47^3 53$
19	282864115826540259457020866955786597206368	$2^5 7^2 13.29.31.37^5 41^6 43^4 47^4 53^2$
20	7711920365637466944268198334152183056187309072	$2^4 7^3 13.37^4 41^7 43^5 47^5 53^3 59$

THEOREM.

$$N_a = D_a \prod_{i=1}^{a} \frac{3i-1}{3i-2}.$$

Proof. This is just the number of C_3-invariant plane partitions. $\square$

Case 3: $[-1,-1,-1,1,1]$. Squarish.

1	0	0
2	1	1
3	0	0
4	4	2^2
5	0	0
6	9	3^2
7	0	0
8	324	$2^2 3^4$
9	0	0
10	9801	$3^4 11^2$
11	0	0
12	2108304	$2^4 3^2 11^4$
13	0	0
14	577344784	$2^4 6007^2$
15	0	0
16	127860595776	$2^6 3^2 47^2 317^2$
17	0	0
18	270136705018896	$2^4 3^4 19^2 24029^2$
19	0	0
20	15213660435694224	$2^4 3^6 7^2 19^2 31^2 277^2$

CONJECTURE. *$N_{2a+1}=0$; N_{2a} =perfect square.*

Case 7: $[-1, 1, 1, -1, -1]$. Squarish.

1	0	0
2	-1	-1
3	0	0
4	0	0
5	0	0
6	-25	-5^2
7	0	0
8	64	2^6
9	0	0
10	-58081	-241^2
11	0	0
12	3779136	$2^6 3^{10}$
13	0	0
14	-10225254400	$-2^{16} 5^2 79^2$
15	0	0
16	10207258214400	$2^{28} 3^2 5^2 13^2$
17	0	0
18	-128076307869270016	$-2^{16} 1397959^2$
19	0	0
20	1624216947480625000000	$2^6 5^{10} 1612063^2$

CONJECTURE. $N_{2a+1} = 0$; $(-1)^a N_{2a}$ =*perfect square.*

Case 16: $[1, 1, 1, -1, -1]$. Squarish.

1	2	2
2	5	5
3	12	$2^2 3$
4	36	$2^2 3^2$
5	132	$2^2 3.11$
6	605	5.11^2
7	3652	$2^2 11.83$
8	27556	$2^2 83^2$
9	275228	$2^2 83.829$
10	3436205	5.829^2
11	57360168	$2^3 3^2 31^2 829$
12	1196883216	$2^4 3^4 31^4$
13	33444368352	$2^5 3^2 31^2 149.811$
14	1168165113680	$2^4 5.149^2 811^2$
15	54801178665792	$2^6 3.13.149.811.181693$
16	3213553833262656	$2^6 3^2 13^2 181693^2$
17	253291522613787840	$2^6 3.5.7^3 13.181693.325667$
18	24955469379391922000	$2^4 5^3 7^6 325667^2$
19	3308750836781311159840	$2^5 5.7^3 137143.325667.1349903$
20	548368372352759739882256	$2^4 137143^2 1349903^2$

CONJECTURE. *N_{4a} and $5N_{4a+2}$ are perfect squares, and $5N_{2a+1}^2 = 4N_{2a}N_{2a+2}$.*

Case 2: $[-1,-1,-1,-1,1]$. Numerological.

1	0	0
2	1	1
3	0	0
4	0	0
5	0	0
6	21	3.7
7	-168	$-2^3 3.7$
8	1872	$2^4 3^2 13$
9	0	0
10	114257	$11.13.17.47$
11	11671792	$2^4 11.17.47.83$
12	-27000896	$-2^6 13.17.23.83$
13	0	0
14	-1361768544000	$-2^8 3^3 5^3 17.23.29.139$
15	-143003331532800	$-2^{11} 3^4 5^2 17.23.29.3041$
16	-19098012740382720	$-2^{12} 3^3 5.17.23.31.937.3041$
17	0	0
18	$-1436383872259945459 20$	$-2^8 3^2 5.7^2 19.23^2 31.19571.41729$
19	320290286404151426146560	$2^8 3^2 5.7^2 19.23^2 31.79.163.3389.41729$
20	37402033607864918373044160	$2^6 3^2 5.7^3 23.31.79.163.3389.6661.182681$

Note that N_{4a+1} appears to be zero, and $\gcd(N_{4a+2}, N_{4a+3})$ and $\gcd(N_{4a-1}, N_{4a})$ appear to be unreasonably large.

Case 8: $[-1, 1, 1, -1, 1]$. Numerological.

1	0	0
2	-1	-1
3	-4	-2^2
4	-12	$-2^2 3$
5	0	0
6	-357	$-3.7.17$
7	4284	$2^2 3^2 7.17$
8	-29148	$-2^2 3.7.347$
9	0	0
10	17523135	$3^4 5.7^2 883$
11	-927891720	$-2^3 3^3 5.7.139.883$
12	43069127760	$2^4 3^3 5.37.139.3877$
13	0	0
14	204836871805104	$2^4 3.7.11.23.199.12108637$
15	45508880439261120	$2^6 3.5.7.17.167.197.12108637$
16	2672995365361742400	$2^6 3^3 5^2 7.17.167.197.15804647$
17	0	0
18	$-638412029586375199122960$	$-2^4 3^3 5.17.23.107.5563.10343.122781007$
19	$-44029760935772646389012112\,0$	$-2^5 3^3 5.7.13.17.10343.51879353.122781007$
20	$-29815634454779934206194080696\,0$	$-2^4 3.5.7^2 13.17.307.51879353.7202968791331$

Note that N_{4a+1} appears to be zero, and $\gcd(N_{4a+2}, N_{4a+3})$ and $\gcd(N_{4a-1}, N_{4a})$ appear to be unreasonably large.

Case 11: $[1, -1, -1, -1, 1]$. Numerological.

1	2	2
2	3	3
3	0	0
4	20	$2^2 5$
5	-60	$-2^2 3.5$
6	231	$3.7.11$
7	0	0
8	-5148	$-2^2 3^2 11.13$
9	180180	$2^2 3^2 5.7.11.13$
10	-2567565	$-3^3 5.7.11.13.19$
11	0	0
12	-2803220784	$-2^4 3^6 7.13.19.139$
13	-141545345760	$-2^5 3^2 5.7.13.19.139.409$
14	-5915647639440	$-2^4 3^3 5.7.11.19.23.199.409$
15	0	0
16	51583259724171840	$2^6 3^2 5.7.17.19^3 23.29.167.197$
17	18924089325965801280	$2^6 3^4 5.17.19.29.167.197.2369173$
18	4053331253074987825200	$2^4 3^4 5^2 7.19.23.29.107.5563.2369173$
19	0	0
20	6817926428328222067622580000	$2^4 3^3 5^3 7^2 13.29.37.51879353.356061359$

Note that N_{4a-1} appears to be zero, and $\gcd(N_{4a}, N_{4a+1})$ and $\gcd(N_{4a+1}, N_{4a+2})$ appear to be unreasonably large.

Case 12: $[1,-1,-1,1,1]$. Numerological.

1	2	2
2	3	3
3	0	0
4	0	0
5	32	2^5
6	-21	-3.7
7	0	0
8	0	0
9	5184	$2^6 3^4$
10	43443	$3^3 1609$
11	-231696	$-2^4 3^2 1609$
12	6029376	$2^6 3.31.1013$
13	32156672	$2^{10} 31.1013$
14	456929280	$2^{12} 3^2 5.37.67$
15	0	0
16	0	0
17	-13159563264000	$-2^{19} 3^4 5^3 37.67$
18	440535780864000	$2^{12} 3^5 5^3 3540829$
19	-34692192743040000	$-2^{10} 3^7 5^4 7.3540829$
20	3970922268778920000	$2^6 3^8 5^4 7^2 13.47.83.6089$

Based on slightly more extensive data than this, zeros of N_a seem not to occur for $a > 16$. Consecutive terms do appear to have unreasonably large gcd's.

Case 17: $[1, 1, 1, -1, 1]$. Numerological.

1	2	2
2	5	5
3	0	0
4	0	0
5	-160	$-2^5 5$
6	-175	$-5^2 7$
7	0	0
8	-129584	$-2^4 7.13.89$
9	2576728	$2^3 7.11.47.89$
10	-66034859	$-11.47.127727$
11	0	0
12	7891156416	$2^6 3.11.13.97.2963$
13	-8238367298304	$-2^8 3^3 11.13.29.97.2963$
14	290534051149056	$2^8 3^3 11^2 13.29.163.5653$
15	0	0
16	19773489658292637696	$2^{12} 3^3 11.13.19.937.70231247$
17	5830680025363675981824	$2^{11} 3^2 13.19^2 23.41729.70231247$
18	1806147811748339411339520	$2^8 3^2 5.13.19.23.41729.661360181149$
19	0	0
20	$-185017961665936020816649713216$	$-2^6 3.7.19.6661.182681.5954265672137891$

Note that N_{4a-1} appears to be zero, and $\gcd(N_{4a}, N_{4a+1})$ and $\gcd(N_{4a+1}, N_{4a+2})$ appear to be unreasonably large.

Case 4: $[-1,-1,1,-1,-1]$. Garbage.

1	0	0
2	1	1
3	-2	-2
4	-4	-2^2
5	-10	-2.5
6	-59	-59
7	282	$2.3.47$
8	1420	$2^2 5.71$
9	16098	$2.3.2683$
10	233941	233941
11	-2805644	$-2^2 353.1987$
12	-37469104	$-2^4 2341819$
13	-1593969936	$-2^4 3.13.2554439$
14	-62160804144	$-2^4 3^2 7639.56509$
15	1919926898208	$2^5 3^2 11^2 83.663787$
16	68829915214016	$2^6 11^2 139.2239.28559$
17	10244839440624800	$2^5 5^2 11.17.769.89052727$
18	1099381927654355536	$2^4 89.109.491.11719.1230949$
19	-88841168986020094320	$-2^4 3.5.21377387.17316032939$
20	-8588578028063343366704	$-2^4 4010004461.133861728079$

See Conjecture 5 and Case 13.

Case 5: $[-1,-1,1,-1,1]$. Garbage.

1	0	0
2	1	1
3	2	2
4	-4	-2^2
5	30	$2.3.5$
6	-139	-139
7	710	$2.5.71$
8	6748	$2^2 7.241$
9	-84206	$-2.71.593$
10	3704689	3704689
11	-110189516	$-2^2 79.239.1459$
12	-2872124016	$-2^4 3.199.300683$
13	-270783829936	$-2^4 31^2 17610811$
14	-22527295095856	$-2^4 31.971.46774391$
15	-1053324013294112	$-2^5 7.19.247491544477$
16	235837385312703424	$2^6 19.23.8432400790643$
17	55026090070639235232	$2^5 3.17.41.1811.12269.37011529$
18	$2665022499485232134747 2$	$2^4 13.3229.39679802324565121$
19	$1190908386577631872830827 2$	$2^4 71.5867.25665911.69618914921$
20	$-44097137250154759881733404 64$	$-2^4 2657.86083.1204984513805801009$

Case 6: $[-1, -1, 1, 1, 1]$. Garbage.

1	0	0
2	1	1
3	-2	-2
4	4	2^2
5	-50	-2.5^2
6	-251	-251
7	-2358	$-2.3^2 131$
8	-72540	$-2^2 3^2 5.13.31$
9	933498	$2.3^3 59.293$
10	-38253699	$-3^2 11.386401$
11	3084522804	$2^2 3^2 11^2 708109$
12	97662336112	$2^4 11.19319.28723$
13	17529913245520	$2^4 5.13.66413.253801$
14	3819970691582672	$2^4 11.37.3347.175262873$
15	-275125042670411232	$-2^5 3^3 13.59.359.479.2414299$
16	229013344926722884160	$2^6 5.29.24678162168827897$
17	$-137442954942027582135008$	$-2^5 7.17.8781547.4110120114083$
18	$-19124026283095423532492336$	$-2^4 19.16618451.3785429893218859$
19	$-85746202736794638798624983376$	$-2^4 3.19^2 174067.28428242585726122901$
20	$-143363820408339238563749597692176$	$-2^4 3.19.859.1553.117836574147578701262699$

Case 13: $[1,-1,1,-1,-1]$. Garbage.

1	2	2
2	-1	-1
3	-2	-2
4	-4	-2^2
5	-18	-2.3^2
6	59	59
7	314	2.157
8	1420	$2^2 5.71$
9	19034	$2.31.307$
10	-233941	-233941
11	-3538620	$-2^2 3^3 5.6553$
12	-37469104	$-2^4 2341819$
13	-1504622256	$-2^4 3.31346297$
14	62160804144	$2^4 3^2 7639.56509$
15	2692463297568	$2^5 3.11.2549681153$
16	68829915214016	$2^6 11^2 139.2239.28559$
17	8276014840508448	$2^5 3.11.107.131.559116449$
18	-1099381927654355536	$-2^4 89.109.491.11719.1230949$
19	$-13650884333283807707 2$	$-2^4 13327.51631.12399320041$
20	$-858857802806334336670 4$	$-2^4 4010004461.133861728079$

See Conjecture 5 and Case 4.

Case 14: $[1,-1,1,-1,1]$. Garbage.

1	2	2
2	-1	-1
3	2	2
4	-12	$-2^2 3$
5	-6	-2.3
6	99	$3^2 11$
7	-2178	$-2.3^2 11^2$
8	23828	$2^2 7.23.37$
9	-423218	$-2.61.3469$
10	-4519177	-13.347629
11	-69590316	$-2^2 3.17.73.4673$
12	-9813606288	$-2^4 3.204450131$
13	195922249008	$2^4 3.4081713521$
14	25827832334128	$2^4 11.37^2 107194337$
15	5584861953489760	$2^5 5.34905387209311$
16	845933404903702848	$2^6 3^2 29.4817.10513299761$
17	186553511067917785440	$2^5 3.5.11.6735307.5245804589$
18	-35644755831028059013776	$-2^4 3^2 12148418287.20375741167$
19	2241946865834283828525744	$2^4 3.17.47.149.2717447.144374318749$
20	$-1455683832282340125375884 1680$	$-2^4 5.13.907.49102152839.314286624329$

Case 15: $[1, -1, 1, 1, 1]$. Garbage.

1	2	2
2	-1	-1
3	6	2.3
4	-28	$-2^2 7$
5	-18	-2.3^2
6	-997	-997
7	-12334	$-2.7.881$
8	-2460	$-2^2 3.5.41$
9	-4705254	$-2.3^2 173.1511$
10	160936899	3.53645633
11	-1258566892	$-2^2 11.31.211.4373$
12	643185178352	$2^4 17.2364651391$
13	61006278023472	$2^4 3.17.74762595617$
14	2963818273276976	$2^4 53.2707.1291122541$
15	2549040699130953376	$2^5 79657521847842293$
16	-666159591454801167808	$-2^6 7.101.122173.120504609377$
17	$15706890784616673236457 6$	$2^5 3^2 113.127.1061.2347.94933.160757$
18	$-2931122451498617558662350 88$	$-2^4 18319515321866359741639693$
19	$-208076361518409478574846986096$	$-2^4 19.684461715521083811101470349$
20	$-2170108833861972096239546113931 68$	$-2^4 135631802116373256014971632120 73$