Math 471 - Introduction to Numerical Methods - Fall 2019

Assignment # 2.
Due: Thursday, September 19, 2019.

1. Consider the function $f(x) = \ln(1+x)$.

   a) Write down the Taylor polynomial of degree n about $x_0 = 0$.

   b) How large must n be in order to approximate $ln(1.1)$ to within 10^{-5}?

2. Consider the function $f(x) = sin(\pi x/2)$.

   a) Expand $f(x)$ in a Taylor series about the point $x_0 = 0$.

   b) Find an expression for the remainder.

   c) Estimate the number of terms that would be required to guarantee accuracy of 10^{-6} for $f(x)$ for all x in the interval [-1, 1]. Write explicitly the resulting Taylor polynomial.

   d) On the same graph, plot $f(x)$ and its 1st, 3rd, 5th and 7th degree Taylor polynomials over [-2, 2]. Discuss what you observe. How accurate are the successive Taylor polynomials in approximating $f(x)$?

3. Let A be the 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Use Gauss elimination to obtain A^{-1} by solving the two systems $Ax_1 = e_1$ and $Ax_2 = e_2$, where e_1 and e_2 are the columns of the 2×2 identity matrix. Note that you can perform both at the same time by considering the augmented system $[A|I]$. Show that A^{-1} exists if and only if $det(A) \neq 0$.

4. Let $E_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -m_{2,1} & 1 & 0 & 0 \\ -m_{3,1} & 0 & 1 & 0 \\ -m_{4,1} & 0 & 0 & 1 \end{pmatrix}$, $E_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -m_{3,2} & 1 & 0 \\ 0 & -m_{4,2} & 0 & 1 \end{pmatrix}$, $P_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

   (a) Show that

   $$E_1^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ m_{2,1} & 1 & 0 & 0 \\ m_{3,1} & 0 & 1 & 0 \\ m_{4,1} & 0 & 0 & 1 \end{pmatrix},$$

   (b) Show that

   $$E_1^{-1}E_2^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ m_{2,1} & 1 & 0 & 0 \\ m_{3,1} & m_{3,2} & 1 & 0 \\ m_{4,1} & m_{4,2} & 0 & 1 \end{pmatrix}.$$

   (c) Show that $P_1^{-1}=P_1$.

5. Find the LU factorization of A and use it to solve $Ax = b$.

$$A = \begin{pmatrix} 4 & 1 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 1 & 4 \end{pmatrix}, \quad b = \begin{pmatrix} 3 \\ -2 \\ 2 \\ -3 \end{pmatrix}.$$

6. Gaussian Elimination: Operation count. Prove that $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$