Math 471 - Introduction to Numerical Methods - Fall 2019

Assignment # 3.
Due: Thursday, September 26, 2019.

1. **Elementary Matrices**

   Let B be a 4×4 matrix to which we apply the following operations.

   - Double column 1,
   - halve row 3,
   - add row 3 to row 1,
   - interchange columns 1 and 4,
   - subtract row 2 from each of the other rows,
   - replace column 4 by column 3,

   (a) Write the result as a product of seven matrices, including B.

   (b) Write the result again as a product of three matrices, including B.

2. **Special Matrices:**

   Consider the matrix

$$\begin{pmatrix} b & -1 & 0 \\ -1 & 4 & 1 \\ 0 & 1 & 5 \end{pmatrix}.$$

   (a) For what values of b is this matrix positive definite?

   (b) For what values of b is this matrix strictly diagonally dominant? Recall that an $n \times n$ matrix A is said to be *strictly diagonally dominant* if

$$\sum_{j=1,\, j\neq i}^{n} |a_{ij}| < |a_{ii}| \qquad \text{for } i = 1, \ldots, n\,.$$

3. **Partial Pivoting**

   (a) Prove that the matrix

$$\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$

   does not have an LU decomposition.

   (b) Does the system

$$\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$

   have a *unique* solution for all $a, b \in \mathbb{R}$? How do you know?

   (c) How can you modify the system in part (b) so that LU decomposition applies?

4. Consider the linear system $Ax = b$ where A is the following matrix,

$$A = \begin{pmatrix} -5 & 2 & -1 \\ 1 & 0 & 3 \\ 3 & 1 & 6 \end{pmatrix}.$$

(a) Using Gaussian Elimination with partial pivoting to reduce the matrix to upper traingular form. Show clearly *each step* Write the corresponding L, U matrices. Multiply the L, U matrices, what do you get? Determine the permutation matrix P such that $PA = LU$.

(b) Use the P, L, U decomposition found in (a) to find the solution to $Ax = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$.

Be sure to show *all* relevant steps.

(c) Use the P, L, U decomposition found in (a) to find the solution to $Ax = \begin{pmatrix} 0 \\ 1 \\ 5 \end{pmatrix}$.

Again, clearly show *all* relevant steps.

5. **Cholesky Factorization**

Find the Cholesky factorization $A = LL^T$ *by hand* for

$$A = \begin{pmatrix} 4 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 4 \end{pmatrix}.$$

6. **Gaussian Elimination (with Partial Pivoting)**

Write a program to solve an $n \times n$ linear system using Gaussian Elimination. As a test of your code, run your code on the system

$$A = \begin{pmatrix} 2 & 2 & -3 \\ 3 & 1 & -2 \\ 6 & 8 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -3 \\ 0 \\ -1 \end{pmatrix}.$$

Extra Credit: Incorporate partial pivoting into your code, test it on the above system.

Print and attach the text file containing your program. No code, no credit.