Math 471 - Introduction to Numerical Methods - Fall 2019

Assignment # 4.
Due: Thursday, October 3, 2019.

1. **Matrix Norms:**

   Suppose that $A \in \mathbb{R}^{n\times n}$ is invertible, B is an approximation of A^{-1}, so that $AB = I + E$. Show that the relative error in B is bounded by $\|E\|$.

2. (a) Consider the matrix,

   $$A = \begin{bmatrix} 2 & -3 & 1 \\ -4 & 1 & 2 \\ 5 & 0 & 1 \end{bmatrix}.$$

   Compute $\|A\|_\infty$ and find a vector x such that $\|A\|_\infty = \|Ax\|_\infty / \|x\|_\infty$.

   (b) Find an example of a 2×2 matrix A such that $\|A\|_\infty = 1$ but $\rho(A) = 0$. This shows that the spectral radius $\rho(A) = \{\max |\lambda| : \ \lambda \text{ is an eigenvalue of } A\}$ **does not** define a matrix norm.

3. Consider the matrix, right side vector, and two approximate solutions,

   $$A = \begin{pmatrix} 1.2969 & 0.8648 \\ 0.2161 & 0.1441 \end{pmatrix}, \quad b = \begin{pmatrix} 0.8642 \\ 0.1440 \end{pmatrix}, \quad x_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad x_2 = \begin{pmatrix} 0.9911 \\ -0.4870 \end{pmatrix}.$$

   (a) Show that $x = (2, -2)^T$ is the exact solution of $Ax = b$.

   (b) Compute the error and residual vectors for x_1 and x_2.

   (c) Find $||A||_\infty, ||A^{-1}||_\infty$ and $\text{cond}_\infty(A)$ (you may use MATLAB for this calculation).

   (d) In class we proved a theorem relating the condition number of A, the relative error, and the relative residual. Check this result for the two approximate solutions x_1 and x_2.

4. **Ill-Conditioning:**

   We revisit Problem 2 of hw0: The $n \times n$ Hilbert matrix H_n, with entries $H_{i,j} = \frac{1}{i+j-1}$, is easily produced in MATLAB using *hilb(n)*. Assume the true solution of $Hx = b$ for a given n is $x = [1, \ldots, 1]^T$, and set up the right hand side vector b accordingly. Solve the system $Hx = b$ for $n = 5, 10, 15, 20$ (you may use the Matlab command $x = H\backslash b$. Extra credit if you use your own code for GE with partial pivoting). For each n, using the $\infty-$norm, compute the relative error and the relative residual. Discuss what is happening here. You may find it useful to look at the *cond* command in MATLAB. **Attach your code**.

5. **Iterative Methods:** Suppose that A is strictly diagonally dominant. Show that the Jacobi iteration matrix satisfies $||M_J||_\infty < 1$ and therefore that Jacobi iteration converges in this case.

   Recall that an $n \times n$ matrix A is said to be *strictly diagonally dominant* if

$$\sum_{j=1,\, j \neq i}^{n} |a_{ij}| < |a_{ii}| \qquad \text{for } i = 1, \ldots, n\,.$$

   Note that the strict inequality implies that each diagonal entry a_{ii} is non-zero. Why do you need to know that?

6. **Iterative Methods:** Given the matrix and right hand side vector,

$$A = \begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix} \;, \; b = \begin{pmatrix} 5 \\ 5 \end{pmatrix}$$

   (a) Perform 3 iterations by hand using Jacobi and Gauss-Seidel methods. Start from initial guess $x_0 = (0,0)^T$.

   (b) Write down the iteration matrices for M_J and M_{GS} and compute the l_∞ norm and spectral radius. Which of the two iterateions converges from arbitrary initial guess? Why?

   (c) Write a program to compute the spectral radius of M_ω for ω values ranging from 0 to 2, in increments of 0.001. Plot $\rho(M_\omega)$ against ω. By inspecting the graph, find the optimal value of the relaxation parameter, ω^*. Compare with the theoretical value discussed in class. **Attach your code and plots**.

   (d) Write a program to solve the linear system numerically, using Jacobi, Gauss-Seidel, and optimal SOR, using $x_0 = (0,0)^T$. Terminate the iteration after step k if $||x^{(k)} - x^{(k-1)}||_\infty < 10^{-5}$. **Attach your code**.

   (e) Make a `semilogy` plot that compares the evolution of the error $||x - x^{(k)}||_\infty$ for all three methods.

   Discuss your results.