Math 471 - Introduction to Numerical Methods - Fall 2019

Assignment # 5.
Due: Thursday, October 10, 2019.

1. **Root Finding:**

   Consider $f(x) = x^3 - 2$.

   (a) Show that $f(x)$ has a root α in the interval $[1, 2]$.

   (b) Compute an approximation to the root by taking 4 steps of the bisection method.

   (c) Repeat, using Newton's method. Take $x_0 = 1.5$ for the starting value.

   For each method, present your results in the form of a table. For the bisection method, tabulate the interval $[a_n, b_n]$, the midpoint x_{n+1}, $f(x_{n+1})$ and the error $|x_{n+1} - \alpha|$. For Newton's method, tabulate x_n, $f(x_n)$ and the error $|x_n - \alpha|$. **Discuss your results.**

2. **Fixed-Point Iteration:**

   Which of the following iterations $x_{n+1} = g(x_n)$ will converge to the indicated fixed point α (provided x_0 is sufficiently close to α)? If it does converge, give the order of convergence; for linear convergence, compute $g'(\alpha)$. In the case that $g'(\alpha) = 0$, expand $g(x)$ in a Taylor polynomial about $x = \alpha$ to determine the order of convergence.

   (a) $x_{n+1} = -16 + 6x_n + \dfrac{12}{x_n}, \qquad \alpha = 2$

   (b) $x_{n+1} = \dfrac{2}{3}x_n + \dfrac{1}{x_n^2}, \qquad \alpha = 3^{1/3}$

   (c) $x_{n+1} = \dfrac{12}{1 + x_n}, \qquad \alpha = 3$

Ill-behaved root-finding:

3. Consider the function $f(x) = \tan(x) - x$.

   (a) Use Newton's method to find the root near $x = 101$. You will observe that this root is difficult to find. Starting with $x_0 = 101$ is not a good initial guess. Use a graphical method to determine roughly where α is, then choose an initial condition x_0 sufficiently close to α in order to achieve convergence. To explain the difficulty compute the quantity $M \approx \frac{1}{2}\frac{|f''(\alpha)|}{|f'(\alpha)|}$, and refer to the discussion in class concerning M. Discuss your findings.

   (b) Reformulate the problem of finding a root of $f(x)$ by finding a function $h(x)$ whose roots are identical to those of $f(x)$ (hint: use the fact that $\tan x = \frac{\sin x}{\cos x}$). Apply Newton's method to the $h(x)$ that you found with $x_0 = 101$. Comment on the convergence in this case as compared to the findings in part a).

4. Solve the equation $x^3 - 3x^2 + 3x - 1 = 0$ using Newton's method with initial guess $x_0 = 1.001$. Discuss the convergence of Newton's method for this problem.

5. In our analysis of Newton's method we showed that if $f'(\alpha) \neq 0$ (i.e. α is a *simple* root), then second order convergence results. However, if α is a *multiple* root of $f(x)$ of multiplicity p then

$$f(\alpha) = f'(\alpha) = f''(\alpha) = ... = f^{(p-1)}(\alpha) = 0\,.$$

In this case, we can write

$$f(x) = (x - \alpha)^p h(x)$$

for some function $h(x)$, and $h(\alpha) \neq 0$.

   (a) Write the iteration function for Newton's method in this case and evaluate $g'(\alpha)$ (note: it will involve $h(x)$ and $h'(x)$).

   (b) What is the rate of convergence of Newton's method in this case?

   (c) Discuss again the convergence of Newton's method in Problem 4.

6. **Root of Nonlinear Systems:**

   Write down Newton's method to solve the system $x^2 + y^2 = 4$, $x^2 - y^2 = 1$. Perform one step of Newton's method with initial guess $x_0 = 1, y_0 = 1$.