Interpolation:

1. The function \( f(x) = e^x \) is given at the 4 points: \( x_0 = 0, x_1 = 1, x_2 = 2, x_3 = 3 \).
   
   (a) Write the interpolating polynomial in Lagrange form.
   
   (b) Write the interpolating polynomial in Newton form.
   
   (c) Evaluate \( e^{1.5} \) and \( e^4 \) using the interpolating polynomial. Which approximate value is more accurate?
   
   (d) Use the error formula to find an upper bound for the maximum error
      \[
      ||f - p_3||_\infty = \max_{1 \leq x \leq 4} |f(x) - p_3(x)|.
      \]

2. The following data are taken from a polynomial \( p(x) \) of degree \( \leq 5 \). What is the actual degree of \( p(x) \)? Explain.

   \[
   \begin{array}{c|cccccc}
   x & -2 & -1 & 0 & 1 & 2 & 3 \\
   p(x) & -5 & 1 & 1 & 7 & 25 & \\
   \end{array}
   \]

3. Show that \( \sum_{k=0}^{n} \ell_k(x) = 1 \) (hint: consider the function \( f(x) = 1 \))

4. Chebyshev polynomials:
   
   The Chebyshev polynomials are defined for \( x \in [-1, 1] \) by \( T_n(x) = \cos(n\theta) \), \( x = \cos \theta \).
   
   (a) Derive the 3-term recurrence relation,
   
   \[
   T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x).
   \]
   
   (b) Given \( T_0(x) = 1 \) and \( T_1(x) = x \), use the recurrence relation to find \( T_2(x) \) and \( T_3(x) \).
   
   (c) What are the roots of \( T_3(x) \)?
5. We want to study the effect of different choices of interpolation points \( \{x_0, x_1, ..., x_n\} \) on the function 
\[
    w_n(x) = (x - x_0)(x - x_1) ... (x - x_n)
\]
in the formula for the error in interpolation polynomials. In particular, we want to study evenly spaced points and Chebyshev points in the interval \([-1, 1]\). Consider the following choices:

(a) \( x_i = -1 + \frac{2i}{n} \quad i = 0...n \)

(b) \( x_i = -\cos \frac{\pi}{n+1}(\frac{1}{2} + i) \quad i = 0...n \).

In each case, plot \( w_{10}(x) \) in the interval \([-1,1]\). Discuss the results.

6. Write a computer program to perform polynomial interpolation using equally spaced points and the Chebyshev points on the interval \([-1,1]\) for the function \( f(x) \). Investigate the convergence of \( p_n \) to \( f \) by running the program for \( n = 8, 16, 32 \) in the following cases

\[
    f_1(x) = |x| , \quad f_2(x) = \begin{cases} 
    -1 & \text{if } x < 0, \\
    0 & \text{if } x = 0, \\
    1 & \text{if } x > 0.
\end{cases}
\]

Discuss the results. As \( n \) gets larger, is there pointwise convergence? Is convergence uniform in \( x \)?

(in MATLAB \( f_1(x) = \text{abs}(x) \) and \( f_2(x) = \text{sign}(x) \), you can use the library function ”polyfit” in MATLAB. Use ”help polyfit” to find how to use it).