Math 471 - Introduction to Numerical Methods - Fall 2019

Assignment # 6
Due: Thursday, October 24, 2019.

Interpolation:

1. The function $f(x) = e^x$ is given at the 4 points: $x_0 = 0, x_1 = 1, x_2 = 2, x_3 = 3$.
   (a) Write the interpolating polynomial in Lagrange form.
   (b) b) Write the interpolating polynomial in Newton form.
   (c) c) Evaluate $e^{1.5}$ and e^4 using the interpolating polynomial. Which approximate value is more accurate?
   (d) Use the error formula to find an upper bound for the maximum error
   $$||f - p_3||_\infty = \max_{1\leq x\leq 4} |f(x) - p_3(x)|\,.$$

2. The following data are taken from a polynomial $p(x)$ of degree ≤ 5. What is the actual degree of $p(x)$? Explain.

x	-2	-1	0	1	2	3
$p(x)$	-5	1	1	1	7	25

3. Show that $\displaystyle\sum_{k=0}^{n} \ell_k(x) = 1$ (hint: consider the function $f(x) = 1$)

4. **Chebyshev polynomials:**
   The Chebyshev polynomials are defined for $x \in [-1, 1]$ by $\mathrm{T}_n(x) = \cos(n\theta)$, $x = \cos\theta$.
   (a) Derive the 3-term recurrence relation,
   $$\mathrm{T}_{n+1}(x) = 2x\mathrm{T}_n(x) - \mathrm{T}_{n-1}(x)\ .$$
   (b) Given $\mathrm{T}_0(x) = 1$ and $\mathrm{T}_1(x) = x$, use the recurence relation to find $\mathrm{T}_2(x)$ and $\mathrm{T}_3(x)$.
   (c) What are the roots of $T_3(x)$?

Error and Convergence:

5. We want to study the effect of different choices of interpolation points $\{x_0, x_1, ..., x_n\}$ on the function $w_n(x) = (x - x_0)(x - x_1)...(x - x_n)$ in the formula for the error in interpolation polynomials. In particular, we want to study evenly spaced points and Chebyshev points in the interval $[-1, 1]$. Consider the following choices:

   (a) $x_i = -1 + \dfrac{2i}{n} \qquad i = 0...n$

   (b) $x_i = -\cos\dfrac{\pi}{n+1}(\dfrac{1}{2} + i) \qquad i = 0...n.$

   In each case, plot $w_{10}(x)$ in the interval [-1,1]. Discuss the results.

6. Write a computer program to perform polynomial interpolation using equally spaced points and the Chebyshev points on the interval [-1,1] for the function $f(x)$. Investigate the convergence of p_n to f by running the program for $n = 8, 16, 32$ in the following cases

$$f_1(x) = |x| \ \ , \text{ and } \quad f_2(x) = \begin{cases} -1 & \text{if } x < 0, \\ 0 & \text{if } x = 0, \\ 1 & \text{if } x > 0. \end{cases}$$

   Discuss the results. As n gets larger, is there pointwise convergence? Is convergence uniform in x?

   (in MATLAB $f_1(x) = abs(x)$ and $f_2(x) = sign(x)$, you can use the library function "polyfit" in MATLAB. Use "help polyfit" to find how to use it).