Math 471 - Introduction to Numerical Methods - Fall 2019

Assignment # 9.
Due: Tuesday, November 19, 2019.

1. (a) Construct the quadrature formula for $\int_0^{2h} f(x)dx$, using the second order interpolating polynomial through the points $x_0 = 0$, $x_1 = h + \varepsilon$, and $x_2 = 2h$ and $\varepsilon \in (-h, h)$.

   (b) To establish the order of accuracy of the method, check its accuracy on polynomials. Since it is based on parablolic interpolation, it should give the *exact* answer for polynomials of degree ≤ 2. Verify that it does. Now take $f(x) = x^3$. Is your method exact? Does your answer depend on ε? Conclude that for any choice of ε other than $\varepsilon = 0$, the method is $O(h^4)$ instead of being $O(h^5)$.

2. (a) Apply the Gram-Schmidt orthogonalization method to find the 4th degree Legendre polynomial $P_4(x)$. The first 3 were derived in class and are:

$$P_0 = 1, \qquad P_1 = x$$

$$P_2 = x^2 - \frac{1}{3}, \qquad P_3 = x^3 - \frac{3}{5}x$$

   (b) Express x^4 as a linear combination of the first four Legendre polynomials $\{P_0, P_1, P_2, P_3, P_4\}$.

3. Consider the integral,

$$\int_0^1 x \exp^{-x^2} dx \ .$$

   (a) Use the 4-point Gaussian quadrature rule to approximate the integral (after changing variables to obtain an integral over $[-1, 1]$).

   The points and weights are:

$$x_1 = -0.861136311594053 \qquad c_1 = 0.347854845137454$$
$$x_2 = -0.339981043584856 \qquad c_2 = 0.652145154862546$$
$$x_3 = -x_2 \qquad c_3 = c_2$$
$$x_4 = -x_1 \qquad c_4 = c_1$$

   (b) What value of n would be needed to obtain the same accuracy if the compound Trapezoid rule were used?

4. We proved in class that Gauss-Legendre Quadrature rule

$$(*) \qquad \int_{-1}^{1} f(x)\,dx \approx \sum_{j=1}^{n} c_j f(x_j)$$

is *exact* for polynomials of degree $\leq 2n$–1, where $\{x_j\}_{j=1}^{n}$ are the n distinct roots of the Legendre polynomial $p_n(x)$ of degree n, and $\{c_j\}_{j=1}^{n}$ the corresponding weights. Show that indeed this is the best we can expect by proving that $(*)$ is *not exact* for

$$f(x) = \prod_{j=1}^{n} (x - x_j)^2 ,$$

a polynomial of degree $2n$. (Hint: Compute the approximation for any n).

5. The first three Laguerre polynomials are $L_0(x) = 1$, $L_1(x) = 1 - x$, and $L_2(x) = x^2 - 4x + 2$.

   (a) Show that these polynomials are orthogonal over the interval $(0, \infty)$ with respect to the weight function $w(x) = e^{-x}$.

   (b) It is easily seen that the roots of $L_2(x)$ are $x_{1,2} = 2 \pm \sqrt{2}$. Using the method of undetermined coefficients, and the fact the 2-point Gauss-Laquerre quadrature rule

   $$\int_{0}^{\infty} f(x) e^{-x}\,dx \approx c_1 f(x_1) + c_2 f(x_2)$$

   is exact for all polynomials of degree ≤ 3, derive the weights $c_{1,2}$. (They are $c_{1,2} = x_{2,1}/4$.)

6. Write a code that uses the 5-point Gaussian Quadrature to integrate a function from -1 to 1. The roots and weights are as follows:

$$\begin{aligned} c_1 &= c_5 = 0.236926885056189, & x_1 &= -x_5 = -0.906179845938664, \\ c_2 &= c_4 = 0.478628670499366, & x_2 &= -x_4 = -0.538469310105683, \\ c_3 &= 128/225, & x_3 &= 0 \end{aligned}$$

   (a) Use your code to evaluate $\displaystyle\int_{-1}^{1} \left(x^8 + 42x^5\right) dx$.

   (b) Use your code to evaluate $\displaystyle\int_{-\frac{1}{2}}^{0} x^6 dx$. (after an appropriate change of variables)

   Verify that in both cases, you get the exact answer (to the precision of the machine).