MATH 572 - Numerical Methods for Scientific Computing II - 2015

Assignment # 2.
Due: Thursday, January 29, 2015.

1. Show that the eigenvectors and eigenvalues of the matrix

$$A = \frac{1}{h^2}\begin{pmatrix} -2 & 1 & & & & \\ 1 & -2 & 1 & & & \\ & 1 & -2 & 1 & & \\ & & & & & \\ & & & 1 & -2 & 1 \\ & & & & 1 & -2 \end{pmatrix}$$

are given by $r_j^k = \sin k\pi jh$ and $\lambda_k = \dfrac{2}{h^2}(\cos k\pi h - 1)$, $k, j = 1, ..., n$.

2. In class we considered the 2-point BVP $u'' = f$, $u'(0) = \sigma$, $u(1) = \beta$, and derived the finite difference approximation

$$\frac{U_{j-1} - 2U_j + U_{j+1}}{h^2} = f_j \qquad j = 1, ..., n-1$$

$$\frac{U_{n-1} - 2U_n}{h^2} = f_n - \frac{\beta}{h^2} \qquad j = n$$

$$(i) \quad \frac{U_1 - U_0}{h} = \sigma \qquad j = 0$$

$$(ii) \quad \frac{U_1 - U_0}{h} = \sigma + \frac{h}{2} f_0 \qquad j = 0$$

I. Compute the Local Truncation Error (LTE) for the interior points.

II. Show that the LTE at $j = 0$ is $O(h)$ for (i) and $O(h^2)$ for (ii).

III. Show that the method is l_2-stable. (To find the e-values of the matrix A^h find first the e-functions of the corresponding differential operator $\partial^2/\partial x^2$, with boundary conditions $u'(0) = u(1) = 0$).

3. L_∞ stability

In class, we discussed the stability of the centered-difference approximation for the BVP $u'' = f$ with Dirichlet BCs, and showed that $E = A^{-1}\tau$, implying $||E|| \leq ||A^{-1}|| ||\tau||$. To prove $O(h^2)$ convergence in L_∞, we need to show that $||A^{-1}||_\infty \leq C$. In this problem, you are asked to fill in the details of the stability proof.

I. Show that $A^{-1}e_j$ extracts the j^{th} column of the matrix A^{-1}.

II. Show that the solution for $Av = e_j$ ($v = A^{-1}e_j$ is the j^{th} column of A^{-1}) is the vector obtained by evaluating the Green's function $hG(x; x_j)$ on the grid points $x_i = ih$, where

$$G(x; \bar{x}) = \begin{cases} (\bar{x} - 1)x & x \leq \bar{x} \\ (x - 1)\bar{x} & x \geq \bar{x} \end{cases}$$

III. Consider the matrix G whose elements are $G_{ij} = hG(x_i; x_j)$. Show that each element of G is bounded by h.

IV. Show that $||A^{-1}||_\infty = ||G||_\infty \leq 1$

4. Consider the 2-point BVP

$$u'' = e^x \qquad u'(0) = u(1) = 0$$

I. Find the exact solution

II. Set up the FD scheme described in problem (2) and use Gaussian Elimination to solve the resulting linear system. Use 20 grid points ($h = 0.05$). Compute the FD solution with discrete boundary condition (i) and (ii). Compare with the exact solution. Plot exact and computed solution on the same graph.

III. Mesh Refinement (convergence): Compute $||E||_\infty$ for a sequence of increasingly refined grids $h = 0.05, 0.025, 0.0125, 0.00625$. Plot $\log(||E||_\infty)$ against $\log(h)$. Do this for both bc (i) and (ii). Discuss the accuracy of your results.

5. Consider the 2-point BVP $u'' = f$ this time with Neumann bc's $u'(0) = \sigma_1$, $u'(1) = \sigma_2$. In class, we discussed the nonuniqueness of solutions to this problem, and argued why solutions may even fail to exist. We saw that the FD approximation results in a singular matrix.

I. What condition do $f(x)$, σ_1 and σ_2 need to satisfy for solutions to exist?

II. How does this condition relate to the requirement that the right hand side vector of the corresponding linear system must lie in the range of the matrix in order for discrete approximations to have solutions?