M572 - Numerical Methods for Scientific Computing II - 2015

Assignment # 3.
Due: Thursday, February 5, 2015.

1. Variable Coefficients

In class we considered a symmetric difference scheme for the BVP with variable coefficients $(k(x)u')' = f$, $u(0) = \alpha$, $u(1) = \beta$.

$$\frac{1}{h^2}\left(k_{j-1/2}U_{j-1} - (k_{j-1/2} + k_{j+1/2})U_j + k_{j+1/2}U_{j+1}\right) = f_j$$

I. Show that the coefficient matrix for this approximation is negative definite provided $k(x) > 0$.

II. Show that when $f = 0$ solutions of the difference scheme satisfy a maximum principle: the value of the numerical solution U_j lies between α and β so the maximum/minimum of U_j occur on the boundary. (hint: show U_j is a convex combination of its neighbours and conclude that solution extrema can only occur on the boundary).

2. Nonlinear Pendulum.

Solve numerically the nonlinear pendulum problem $\theta'' = -\sin\theta$, with bc's $\theta(0) = \theta(2\pi) = 0.7$. Set up the discrete nonlinear system and solve by Newton's method. Start with initial guesses
   (i) $\theta^{(0)} = 0.7$,
   (ii) $\theta^{(0)} = 0.7\cos t + 0.5\sin t$.
Confirm the method converges quadratically.

   (iii) On physical grounds, this problem does not have a unique solution. Try to find another solution. Find an initial guess $\theta^{(0)}$ that converges to this solution.

3. Elliptic equations.

I. Find the coefficient matrix A for the 5-point Laplacian operator in red-black ordering.

II. Show that the e-vectors and e-values of the 5-point Laplacian are

$$(r^{p,q})_{i,j} = \sin(p\pi ih)\sin(q\pi jh),$$
$$\lambda^{p,q} = \frac{2}{h^2}\left((\cos(p\pi h) - 1) + \cos(q\pi h) - 1\right).$$

4. Singular perturbations and boundary layers.

Consider the BVP $-\epsilon U'' + U = 1$, $U(0) = U(1) = 0$.

I. Find the exact solution.

II. Compute the solution numerically for $\epsilon = 0.001$. Use $h = 0.01$. Compare with exact solution. Discuss the accuracy of your results. Where does the maximum error occur?

III. What type of grids might be more appropriate for problems involving boundary or internal layers? Why? Design a discretization scheme on irregular grids for this problem. Design a numerical grid better suited for the above problem.

IV. BONUS: Compute the numerical solution on this grid, compare with exact solution and discuss accuracy.