M572 - Numerical Methods for Scientific Computing II - 2015

Assignment # 4.
Due: February 17, 2015.

1. Consider the linear system

$$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

I. Set up the iteration matrices, G_J and G_{GS}. Compute $\rho(G)$ and $||G||_2$.

II. Perform 3 iterations by hand for each method. Compute the error in each solution component. Compute the ratio $||e^{(n+1)}||_\infty/||e^{(n)}||_\infty$. (Set up a table with n, $x_1^{(n)}$, $x_2^{(n)}$, $e_1^{(n)}, e_2^{(n)}$, $||e^{(n)}||_\infty$ and $||e^{(n+1)}||_\infty/||e^{(n)}||_\infty$). Discuss the rate of convergence.

III. Set up the iteration matrix for SOR. Show that if $\rho(G) < 1$, then $0 < \omega < 2$.

IV. Compute $\rho(G)$ as a function of ω (You may use Matlab to compute $max|\lambda_k(\omega)|$). Plot $\rho(G)$ vs. ω and find ω_{opt}. Compare with the theoretical prediction.

V. Perform 3 SOR iterations by hand using ω_{opt}. Compute the error. Compute the ratio $||e^{(n+1)}||_\infty/||e^{(n)}||_\infty$. (Set up a table with n, $x_1^{(n)}$, $x_2^{(n)}$, $e_1^{(n)}$, $e_2^{(n)}$, $||e^{(n)}||_\infty$ and $||e^{(n+1)}||_\infty/||e^{(n)}||_\infty$). Discuss the rate of convergence.

2. Show that if A is strictly diagonally dominant, $||G_J|| < 1$ (in which norm?), and therefore the iteration converges.

3. Aitken's acceleration. Assume that the iteration matrix G has a complete set of e-vectors, r_k, with corresponding e-values λ_k. Assume that λ_1 is the largest in magnitude. The set of e-vectors can be used as a basis.

I. By expanding the initial error $e^{(0)}$ in terms of the basis r_k, show that $e^{(n+1)} \approx \lambda_1 e^{(n)}$.

II. From the above, one has

$$\begin{aligned} e^{(n+1)} &\approx \lambda_1 e^{(n)} \\ e^{(n)} &\approx \lambda_1 e^{(n-1)}. \end{aligned}$$

Eliminate λ_1 between the above two (approximate) equations. Replace $e^{(n)} = x - x^{(n)}$ etc. and rearrange the resulting equations to show that the exact solution $x = (x_1, ..., x_n)^T$ satisfies

$$x_i \approx \frac{x_i^{(n+1)}x_i^{(n-1)} - (x_i^{(n)})^2}{x_i^{(n+1)} - 2x_i^{(n)} + x_i^{(n-1)}} \approx x_i^{(n+1)} - \frac{(x_i^{(n+1)} - x_i^{(n)})^2}{x_i^{(n+1)} - 2x_i^{(n)} + x_i^{(n-1)}}$$

III. Consider the linear system

$$\begin{pmatrix} -2.2 & 1 & 1 \\ 0.8 & -2.2 & 1 \\ 1 & 0.9 & -2.1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0.2 \\ 0.4 \\ 0.2 \end{pmatrix}.$$

Start with an initial guess $x^{(0)} = (0, 0, 0)^T$ and perfom 3 Gauss-Seidel iterations. Use Aitken's acceleration on these three iterates to obtain an improved approximations.

IV. Write a small routine for Gauss-Seidel's method and check how many GS iterations would be required to achieve the same accuracy.