M572 - Numerical Methods for Scientific Computing II - 2015

Assignment # 7.
Due: Tuesday, April 7, 2015.

1. Given the discrete sequence $u_j,\ j = 0, \pm 1, \pm 2, ...$ define the Fourier transform $\hat{u}(\xi h)$ by
$$\hat{u}(\xi h) = \frac{h}{2\pi} \sum_{j=-\infty}^{\infty} u_j e^{-ij\xi h}.$$
Verify formally (i) the inverse transform formula and (ii) Parseval's identity
$$\text{(i) } u_j = \int_{-\pi/h}^{\pi/h} \hat{u}(\xi h) e^{ij\xi h} d\xi,$$
$$\text{(ii) } \int_{-\pi/h}^{\pi/h} |\hat{u}(\xi h)|^2 d\xi = \frac{h}{2\pi} \sum_{j=-\infty}^{\infty} |u_j|^2 \ .$$

2. Consider the Crank-Nicolson scheme for the heat equation
$$\frac{u_j^{n+1} - u_j^n}{k} = \frac{1}{2}(D_+ D_- u_j^n + D_+ D_- u_j^{n+1}).$$
I. Compute the LTE.

   II. Show that if $r = k/h^2 \leq 1$ the scheme satisfies a maximum principle $|u_j^{n+1}| \leq \max_j |u_j^n|$.

   III. Show that if $r \leq 1$ the method converges.

3. Consider the following difference scheme for the heat equation
$$\frac{u_j^{n+1} - u_j^n}{k} = D_+ D_- u_j^n - \frac{h^2}{12}(D_+ D_-)^2 u_j^n.$$
I. Show that the LTE is $O(k + h^4)$.

   II. Find the amplification factor $g(\xi h)$.

   III. For what values of $r = k/h^2$ is the scheme stable in the l_2 norm?

4. Consider the following scheme for the heat equation

$$\frac{u_j^{n+1} - u_j^n}{k} = D_+ D_- u_j^{n+1},$$

corresponding to the backward Euler in time and 2nd order centered differencing in space.

I. Use the energy method to prove the scheme is unconditionally stable in the l_2 norm.

II. Find the amplification factor $g(\xi h)$ and show that $|g(\xi h)| \leq 1$ for all $|\xi h| \leq \pi$.

5. Show that

$$u_{i,j}^{n+1} = u_{i,j}^n + k\left(D_+^x D_-^x + D_+^y D_-^y\right) u_{i,j}^n$$

is stable in $l_\infty \iff r = k/h^2 \leq 1/4$, that is show that

$$\| u^n \|_\infty \leq \| u^0 \|_\infty \quad \iff \quad r \leq 1/4$$

6. In class, we proposed the following split-step method as a first attempt to alleviate the computational cost of Crank-Nicolson

(i) $\left(I - \frac{k}{2} D_+^x D_-^x\right) u_{i,j}^* = \left(I + \frac{k}{2}\left(D_+^x D_-^x + D_+^y D_-^y\right)\right) u_{i,j}^n$

(ii) $\left(I - \frac{k}{2} D_+^y D_-^y\right) u_{i,j}^{n+1} = u_{i,j}^*$

Show that the method is unconditionally stable in l_2. By eliminating u^* between the two half-steps, and comparing the resulting expression to the unsplit Crank-Nicolson, determine the order of accuracy of this scheme.

7. Compute the solution of $u_t = \epsilon u_{xx}$, $\epsilon = 0.05$, on $[0, 1]$, with boundary conditions $u(0, t) = u(1, t) = 0$ and initial conditions (i) $u(x, 0) = \sin \pi x$; (ii) $u(x, 0) = 1 - 2|x - \frac{1}{2}|$. Use second order centered differencing approximation in space and (i) forward Euler and (ii) backward Euler schemes in time. Take $h = 0.05$, and $k = 0.1,\ 0.05,\ 0.025, 0.01$. Plot the computed and exact solutions at $t = 0,\ 0.25,\ 0.5,\ 1.0,\ 2.0$. Discuss the results. (If you are using Matlab, use the command subplot to fit several pictures on the same page).