

## M572 - Numerical Methods for Scientific Computing II - 2015

Assignment # 8.

Due: Thursday, April 16, 2015.

1. Consider the Wave equation

$$u_{tt} = a^2 u_{xx}, \quad u(x, 0) = f(x), \quad u_t(x, 0) = g(x).$$

I. Define  $u_1 = u_x$ ,  $u_2 = -u_t$  and transform the wave equation and initial conditions into a first order system  $U_t + AU_x = 0$  in  $U = (u_1, u_2)^T$ .

II. Find the eigenvalues and eigenvectors of  $A$ , and establish that the system is hyperbolic.

III. Solve the linear hyperbolic system for  $u_1(x, t) = u_x(x, t)$  and  $u_2(x, t) = u_t(x, t)$ .

IV. Integrate  $u_1(x, t) = u_x(x, t)$  to obtain D'Alembert formula

$$u(x, t) = \frac{1}{2}(f(x + at) + f(x - at)) + \frac{1}{2a} \int_{x-at}^{x+at} g(s) ds$$

What is the domain of dependence of the solution  $u(x, t)$ ?

2. Consider the linear advection equation  $u_t + au_x = 0$ , where the constant  $a$  may be either positive or negative. The Lax-Friedrichs scheme is

$$\frac{u_j^{n+1} - \frac{1}{2}(u_{j+1}^n + u_{j-1}^n)}{k} + a \frac{u_{j+1}^n - u_{j-1}^n}{2h} = 0$$

I. Show that the scheme is stable in the 2-norm and the  $\infty$ -norm if  $|\nu| < 1$ .

II. Show that the scheme converges with first order accuracy if  $|\nu| < 1$

3. The centered difference scheme

$$\frac{u_j^{n+1} - u_j^n}{k} + a \frac{u_{j+1}^n - u_{j-1}^n}{2h} = 0$$

is unconditionally unstable. One way to stabilize it is by 'smoothing', that is replacing the  $u_j^n$  term in the time derivative by the average of its neighbours  $\frac{1}{2}(u_{j+1}^n + u_{j-1}^n)$ , which gives the Lax-Friedrichs scheme. Another way is to

combine the centered differencing spatial approximation with a different time integration scheme, that is better suited for imaginary eigenvalues. Analyze the accuracy and stability of the resulting scheme.

4. Show that the upwind scheme for  $u_t + au_x = 0$ ,  $a > 0$  is dissipative of order 2 if  $\nu < 1$ .

5. For the linear advection equation ( $a > 0$ ), the exact value of the solution  $u_j^{n+1}$  can be obtained by tracing the characteristic back in time to  $t_n$  and use the fact that the solution remains constant along this line. In general, the characteristic will not cross at grid points but rather in between grid points. This calls for interpolation. Show that linear interpolation of the data at time  $t_n$  (together with characteristic theory) gives the upwind scheme. Show that quadratic interpolation gives the Lax-Wendroff scheme. In both cases, identify special values of the CFL number  $\nu$  for which the interpolation error vanishes (i.e. the numerical solution becomes exact).

6. Consider  $u_t + u_x = 0$  with initial data  $u(x, 0)$  given by

$$f_1(x) = \begin{cases} 1.0 & 0 \leq x < 2 \\ 0.5 & x = 2 \\ 0.0 & 2 < x \leq 6 \end{cases}, \quad f_2(x) = \begin{cases} 0.0 & 0 \leq x < 1 \\ 1 - |x - 2| & 1 \leq x \leq 3 \\ 0.0 & 3 < x \leq 6 \end{cases}.$$

Compute the solution for  $0 \leq x \leq 6$  and  $t = 2$ . Use the upwind, Lax-Friedrichs and Lax-Wendroff schemes with  $h = 0.05$  and  $k = 0.04, 0.06$ . For each scheme, plot the numerical solution and the exact solution at time  $t = 2$ . Discuss the numerical results. Derive the modified equation for the Lax-Wendroff scheme and use it to explain the numerical results.