1. Show that the viscous Burgers’ equation

\[ u_t + uu_x = \epsilon u_{xx} \]

has a travelling wave solution of the form \( u^\epsilon(x,t) = w(x - st) \) by deriving an ODE for \( w \) and verifying that this ODE has a solution of the form

\[ w(y) = U_R + \frac{1}{2}(U_L - U_R) \left[ 1 - \tanh \left( \frac{U_L - U_R}{4\epsilon} y \right) \right] \]

where \( s = \frac{U_L + U_R}{2} \). Sketch the solution and indicate how it varies with \( \epsilon \).

2. Solve Burgers equation \( u_t + uu_x = 0 \) with initial data

\[ u_0(x) = \begin{cases} 
2 & x < 0 \\
1 & 0 < x < 2 \\
0 & 2 < x 
\end{cases} \]

Sketch the characteristics and shock front in the \( x - t \) plane.

3. Consider the traffic flow model derived in class

\[ \rho_t + (V \rho (1 - \rho))_x = 0 \]

(i) Use this model to study the traffic flow problem with initial condition

\[ \rho_0(x) = \begin{cases} 
1 & x < 0 \\
1 - x & 0 \leq x \leq 1 \\
0 & 1 > x 
\end{cases} \]

(ii) Use this model to study the traffic flow after a car stopped to drop off a passenger and then continued. Take \( V = 5 \) (car lengths/second) and use the initial condition

\[ \rho_0(x) = \begin{cases} 
\frac{1}{4} & |x| > 100 \\
1 - \frac{3|x|}{400} & |x| \leq 100 
\end{cases} \]
4. Consider Burgers equation with the following initial data

\[(i)\quad u_0(x) = \begin{cases} 1 & x < 0 \\ 0.1 & x > 0 \end{cases} \quad (ii)\quad u_0(x) = \begin{cases} 0.1 & x < 0 \\ 1 & x > 0 \end{cases} \]

\[u^n_j \approx u(j \Delta x, n \Delta t)\] denotes the numerical solution at \(x = j \Delta x\) and \(t = n \Delta t\). Consider the following numerical schemes

\[(i)\quad \frac{u^n_{j+1} - u^n_j}{\Delta t} + u^n_j \left( \frac{u^n_j - u^n_{j-1}}{\Delta x} \right) = 0 \]

\[(ii)\quad \frac{u^{n+1}_j - u^n_j}{\Delta t} + \frac{u^n_{j-1} + u^n_j}{2} \left( \frac{u^n_j - u^n_{j-1}}{\Delta x} \right) = 0 \]

Compute the solution on \([-1, 1]\) at time \(t = 1\) using 100, 200, 400 and 800 grid points. Boundary conditions: Do not update the boundary values, keep them as given by the initial data. Compare the numerical and exact solutions and determine if your approximation improves with grid refinement. **Note:** You must choose your time step so that the CFL condition is satisfied \(\max_j \left| u^n_j \frac{\Delta x}{\Delta t} \right| < 1\).

5. In class, we have discussed the effect of nonlinear transformations on weak solutions of hyperbolic conservation laws \(u_t + f(u)_x = 0\). Solve \((i)\quad u_t + (\frac{1}{2}u^2)_x = 0\) and \((ii)\quad (u^2)_t + (\frac{2}{3}u^3)_x = 0\) for the same initial data as in problem 4 using the upwind scheme

\[u^{n+1}_j = u^n_j - \Delta t \frac{\Delta x}{\Delta x} (f^n_j - f^n_{j-1}) \]

Compute the solution on \([-1, 1]\) at time \(t = 1\). Use 200 grid points. Discuss your results.