1. Write the wave equation \( \phi_{tt} = c^2 \phi_{xx}, \phi(x,0) = f(x), \phi_t(x,0) = g(x) \) as a first order system and use characteristic decoupling to solve it.

2. The Euler equations expressed in terms of the conserved variables \( W = (\rho, \rho u, E) \) were given in class

\[
\begin{pmatrix}
\rho \\
\rho u \\
E
\end{pmatrix}_t + \begin{pmatrix}
\rho u \\
\rho u^2 + p \\
u(E + p)
\end{pmatrix}_x = 0,
E = \frac{p}{\gamma - 1} + \frac{1}{2} \rho u^2 : \text{EOS for ideal gas}
\]

(i) Compute the Jacobian matrix \( A(W) = \frac{\partial F}{\partial W} \). To confirm hyperbolicity, we need to compute the e-values of \( A(W) \). Considering how messy \( A(W) \) is, this is not a pleasant task. Instead ...

(ii) Use the above equations together with the EOS to derive the Euler equations in terms of the so-called primitive variables \( \tilde{W} = (\rho, u, p) \). This is a quasi-linear system of the form

\[
\tilde{W}_t + A(\tilde{W})\tilde{W}_x = 0.
\]

(iii) Find the e-values of \( A(\tilde{W}) \). Why are the e-values of \( A(\tilde{W}) \) and \( A(W) \) the same? How are the respective sets of e-vectors related?

3. Compute the Hugoniot+rarefaction curves for the Isentropic Euler equations

\[
\begin{pmatrix}
\rho \\
\rho u \\
E
\end{pmatrix}_t + \begin{pmatrix}
\rho u \\
\rho u^2 + p \\
u(E + p)
\end{pmatrix}_x = 0,
p = A\rho^{\gamma}, \quad A : \text{const.}
\]

Note: In the isentropic case, \( p = p(\rho) \) only. Also, recall that \( c = \sqrt{\frac{dp}{d\rho}|_{s}} \) is the speed of sound.

Plot the Hugoniot+rarefaction curves through the point \( \hat{U} = (\hat{\rho}, \hat{m}) = (1, 1) \). (Take \( A = 1, \gamma = 1.4 \)).
4. Consider the Hugoniot+rarefaction curves of the Isothermal Euler equations derived in class

\[ m = \rho \frac{\dot{\rho}}{\rho} - c \sqrt{\frac{\rho}{\rho}} (\rho - \dot{\rho}) : S_1 \]

\[ m = \rho \frac{\dot{\rho}}{\rho} + c \sqrt{\frac{\rho}{\rho}} (\rho - \dot{\rho}) : S_2 \]

\[ m = \rho \frac{\dot{\rho}}{\rho} - c \rho \log \left( \frac{\rho}{\dot{\rho}} \right) : R_1 \]

\[ m = \rho \frac{\dot{\rho}}{\rho} + c \rho \log \left( \frac{\rho}{\dot{\rho}} \right) : R_2 \]

(i) Show that along a rarefaction curve, \( m'(\dot{\rho}) = \lambda_k(\dot{U}) \) and explain why it should be so.

(ii) Show that \( m''(\dot{\rho}) \) is the same along the Hugoniot and the rarefaction curves.

5. Consider the Isothermal Euler equations with \( c = 1 \), and the initial data

\[ U(x, 0) = \begin{cases} 
U_1 & \text{if } x < 0.5 \\
U_2 & \text{if } 0.5 < x < 1 \\
U_3 & \text{if } 1 < x 
\end{cases} \]

\[ U_1 = \begin{pmatrix} \rho_1 \\ m_1 \end{pmatrix} = \begin{pmatrix} 0.28260 \\ 0.098185 \end{pmatrix}, \quad U_2 = \begin{pmatrix} 0.2 \\ 0 \end{pmatrix}, \quad U_3 = \begin{pmatrix} 0.35 \\ -0.19843 \end{pmatrix}. \]

(i) Determine the solution \( U(x, t) \).

(ii) Use the structure of the Hugoniot+rarefaction curves to show that

(a) when 2 shocks of different families collide, the result is 2 shocks.

(b) when 2 shocks of the same family collide, the result is a shock in the same family and a rarefaction in the other.