Arithmetic Aspects of the Theta Correspondence and Periods of Modular Forms

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Summary. We review some recent results on the arithmetic of the theta correspondence for certain symplectic-orthogonal dual pairs and some applications to periods and congruences of modular forms. We also propose an integral version of a conjecture on Petersson inner products of modular forms on quaternion algebras over totally real fields.

1 Introduction

The theta correspondence provides a very important method to transfer automorphic forms between different reductive groups. Central to the theory is the important notion of a dual reductive pair. This is a pair of reductive subgroups G and G' contained in an ambient symplectic group H that happen to be the centralizers of each other in H. In such a situation, for every choice of additive character ψ of $\mathbb{A}/\mathbb{Q}$ and for automorphic representations π, π' on G, G' respectively, one may define theta lifts $\Theta(\pi, \psi), \Theta(\pi', \psi)$ that (if nonzero) are automorphic representations on G', G respectively ([13]). In the automorphic theory, it is an important and subtle question to characterize when the lift is non-vanishing. For instance, the non-vanishing could depend on both local conditions (compatibility of ε-factors) and global conditions (non-vanishing of an L-value).

The theta lift has its genesis in the Weil representation of $H(\mathbb{A})$ on a certain Schwartz space $\mathcal{S}(\mathbb{A})$. For any choice of Schartz function $\varphi \in \mathcal{S}(\mathbb{A})$ and vector $f \in \pi$ one may consider the theta lift $\theta(f, \varphi, \psi)$ which is an element of $\Theta(\pi, \psi)$. Now it is often the case that one can define good notions of arithmeticity for elements of π and $\Theta(\pi, \psi)$. Arithmeticity here could mean algebraicity, rationality over a suitable number field or even p-adic integrality. The main problems in the arithmetic theory of the theta correspondence are the following:

Question A: Suppose f is chosen to be arithmetic. For a given canonical choice of φ, is $\theta(f, \varphi, \psi)$ arithmetic (perhaps up to some canonical transcendental period)?

This question has been studied in great detail by Shimura ([23], [24], [25], and [26]), Harris ([5]), and Harris–Kudla ([6] and [7]) in the case of algebraicity and in some cases rationality over suitable number fields. However, the study of such questions in the setting of p-adic integrality is more recent. Before we mention the progress made recently on this subject, we point out that if the answer to question A is affirmative, one may pose the following:

Question B: Suppose that the form f is a p-unit (with respect to some suitable p-adic lattice.) Is $\theta(f, \varphi, \psi)$ a p-unit ? If not, what can be said about the primes p for which $\theta(f, \varphi, \psi)$ has positive p-adic valuation?

Question B is undoubtedly more difficult than Question A and the answer seems to involve certain kinds of congruences of modular forms and μ-invariants of p-adic L-functions. It is also closely related to the classical question of whether certain spaces of modular forms are (integrally) spanned by theta series.

To the authors knowledge, the only known results on Questions A and B in the integral setting are for the following dual pairs.

(i) $(\mathrm{GU}(2), \mathrm{GU}(3))$ ([3])

(ii) $(\mathrm{GL}(2), \mathrm{GO}(B))$ for B a quaternion algebra. (See [18] for the indefinite case with square-free level over $\mathbb{Q}$, work of Emerton [2] for the definite case at prime level over $\mathbb{Q}$, and Hida [11] for the definite case at full level over totally real fields.)

(iii) $(\mathrm{U}(n), \mathrm{U}(n+1))$ ([8] and [9])

(iv) $(\widetilde{\mathrm{SL}(2)}, \mathrm{O}(V))$, for V the space of trace 0 elements in an indefinite quaternion algebra over $\mathbb{Q}$. This case and applications are treated in forthcoming work of the author ([15], [16], and [17].)

In all these cases, there seem to be intimate connections with Iwasawa theory. For instance, (ii) and (iv) use crucially the main conjecture of Iwasawa theory for imaginary quadratic fields, which is a deep theorem of Rubin. The work of Harris, Li and Skinner has as an application the construction of p-adic L-functions for unitary groups and one divisibility of an associated main conjecture. It is certainly to be expected that other cases of the theta correspondence will yield other applications to Iwasawa theory. In addition to this, one also discovers interesting applications to the study of special values of L-functions and integral period relations for modular forms, on which more will be said later.

In this article, we will focus only on cases (ii) and (iv), mainly out of the author's lack of knowledge of the other cases. Here is a brief outline. We begin by describing some questions regarding periods and congruences of modular forms that motivate the study of arithmeticity of the theta correspondence. Next we explain in some detail the integrality of the Jacquet–Langlands correspondence, i.e., the dual pair $(\mathrm{GL}_2, \mathrm{GO}(B))$, in both the indefinite and the definite setting. Some of the results in the definite setting are new in that they do not seem to have appeared elsewhere. This is followed by a brief discussion of integrality results for the Shimura–Shintani–Waldspurger correspondence, i.e. the dual pair $(\widetilde{\mathrm{SL}_2}, PB^\times)$. Here, surprisingly, the results are more complete in the indefinite case. Finally, we propose a conjecture for the Petersson inner products of modular forms on quaternion algebras over totally real fields. Such a conjecture was first made by Shimura up to algebraic factors and mostly proved by Harris in [5]. Ours is a more refined version up to p-adic units that is motivated by Shimura's conjecture and a computation for elliptic curves over $\mathbb{Q}$.

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Note and caution: In order to keep the exposition simple, we will ignore many terms in the formulas that appear below. For instance, we ignore powers of π (3.1415...), other explicit constants, abelian L-functions, etc. Since we will be interested mostly in p-integrality, we use the symbol $\sim$ instead of $=$ to denote equality up to elements that are units at all places above p. The reader may rest assured that every formula that occurs below may be worked out precisely, so that $\sim$ may be replaced by $=$ after throwing in the appropriate constants and terms that we have neglected in the present exposition.

2 Periods of modular forms

Let f be a holomorphic newform of weight $2k$ on $\Gamma_0(N)$ and K_f the field generated by its Hecke eigenvalues. We assume that N is square-free and that we have picked a factorization $N = N^+N^-$ such that N^- is the product of an even number of primes. Let B be the indefinite quaternion algebra over $\mathbb{Q}$ ramified precisely at the primes dividing N^- and g the Jacquet–Langlands lift of f to the Shimura curve X of level N^+ coming from B. We assume that $p \nmid N$ and normalize g (up to a p-adic unit in K_f) using the integral structure provided by sections of the relative dualizing sheaf on the minimal regular model of X at p. Let F be a field containing K_f and if $2k > 2$ we also assume that B splits over F. It is possible then to attach to f and g certain canonical periods $u_\pm(f), u_\pm(g)$ that are well-defined up to p-adic units in F. (See [15] for instance for a definition.) The usual Petersson inner product is related to these periods by

$$\langle f, f\rangle \sim \delta_f \cdot u_+(f) \cdot u_-(f)$$
$$\langle g, g\rangle \sim \delta_g \cdot u_+(g) \cdot u_-(g)$$

for some algebraic numbers δ_f, δ_g. In fact, one expects (and can show under certain hypotheses) that δ_f (resp. δ_g) is a p-integer that "counts" congruences between f and other eigenforms on $X_0(N)$ (resp. between g and other eigenforms on X).

Remark 2.1. In order to make the last statement precise, one needs to define an invariant attached to (f, λ) that measures congruences of f modulo λ. Let R be the ring of integers of a finite extension of $\mathbb{Q}_p$ containing all the Hecke eigenvalues of all forms of level N, set $\mathbb{T}_R = \mathbb{T} \otimes R$ where $\mathbb{T}$ is the usual Hecke algebra over $\mathbb{Z}$ and let $\mathfrak{m}$ be the maximal ideal of $\mathbb{T}_R$ corresponding to the mod λ representation associated to f. If $\mathbb{T}_\mathfrak{m}$ denotes the localization of $\mathbb{T}_R$ at $\mathfrak{m}$, $\varphi : \mathbb{T}_\mathfrak{m} \to R$ is the eigencharacter of $\mathbb{T}_\mathfrak{m}$ corresponding to f and $\wp$ is the kernel of φ, one defines $\eta_f = \varphi(\mathrm{Ann}(\wp))$. One always has $\eta_f \subseteq (\delta_f)$ as ideals in R based upon the theorem of Hida; and under suitable conditions (such as the freeness of certain cohomology groups as $\mathbb{T}_\mathfrak{m}$-modules) one has also $\eta_f = (\delta_f)$. (For these results see [10] and the references therein as well as Lemma 4.17 of [1].) The reader may easily convince himself/herself that $l(R/\eta_f)$ is a good measure of congruences satisfied by f. Likewise, one may associate to g an invariant η_g using the ring $\mathbb{T}'_{\mathfrak{m}'}$, where $\mathbb{T}'$ is the Hecke algebra for B and $\mathfrak{m}'$ is the maximal ideal corresponding to (g, λ). Again, one always has $\eta_g \subseteq (\delta_g)$ and under suitable freeness assumptions, one has $\eta_g = (\delta_g)$.

Remark 2.2. The Jacquet–Langlands corespondence implies that $\mathbb{T}'$ is a quotient of $\mathbb{T}$, hence $\eta_f \subseteq \eta_g$ as ideals in R. As a consequence, one sees that under suitable conditions, $\delta_g | \delta_f$ and further, the ratio δ_f/δ_g counts congruences between the Hecke eigencharacter associated to f and other systems of eigenvalues that do not transfer to B via the Jacquet–Langlands correspondence. The example below shows that one may expect $u_\pm(f)/u_\pm(g)$ to be a p-unit if p is not an Eisenstein prime for f. This leads to the expectation that

$$\frac{\langle f, f \rangle}{\langle g, g \rangle} = \frac{\delta_f}{\delta_g}$$

up to Eisenstein primes.

Example 2.3. If $2k = 2$ and $K_f = \mathbb{Q}$, we may pick $F = \mathbb{Q}$. Then f and g correspond to elliptic curves E and E' over $\mathbb{Q}$ that are *strong elliptic curves* for $X_0(N)$ and X respectively, i.e., E, E' are realized as quotients $J_0(N) \to E, Jac(X) \to E'$ with the corresponding dual maps being injective. In this case, $u_\pm(f), u_\pm(g)$ agree with the usual periods of E, E' respectively.

Suppose p is not an Eisenstein prime for f, i.e., the mod p Galois representation associated to E is irreducible. Then, by Faltings' isogeny theorem, one may find an isogeny $E \to E'$ of degree prime to p. It follows that $u_\pm(f) \sim u_\pm(g)$, hence $\frac{\langle f,f \rangle}{\langle g,g \rangle} \sim \frac{\delta_f}{\delta_g}$.

As explained in the preceding remark, the number δ_f/δ_g counts congruences between f and forms that do not transfer to the quaternion algebra

B. In fact, using a result of Ribet and Takahashi [20], one can show more precisely in this case that

$$\frac{\langle f, f \rangle}{\langle g, g \rangle} \sim \prod_{q|N^-} c_q$$

where c_q is the order of the component group of the Neron model of E at q (See [18], Section 2.2.1.) Further one knows that the term c_q counts exactly level-lowering congruences at q, i.e., congruences between f and other forms of level dividing N/q.

Remark 2.4. Our motivation lies in proving such results for forms of arbitrary weight. The difficulty is that one does not know how to geometrically relate the motives associated to modular forms of higher weight and those associated to their quaternionic analogues. As the reader will see, the solution we have in mind to this problem is to use automorphic methods to replace the geometric arguments of the example above.

We now make the following assumptions for the rest of this article:

Assumption I: $p > 2k + 1$.
Assumption II: $p \nmid \tilde{N} := \prod_{q|N} q(q+1)(q-1)$.

It can be shown that Assumption II implies in particular that the Condition (*) below is satisfied by p. (See [18], Lemma 5.1.) That p satisfies this condition is essential in order to apply the integrality criteria (3.5) and Proposition 3.9 below.

Condition (*) There exist infinitely many imaginary quadratic fields K that satisfy any prescribed set of splitting conditions at the primes dividing N, are split at p and have class number prime to p.

Question 2.5. Let p be a prime not dividing N. Does p satisfies Condition (*) even if it does not satisfy Assumption II?

3 Arithmeticity of theta lifts

3.1 The pair $(\mathbf{GL_2}, \mathbf{GO}(B))$

In this section, B is a quaternion algebra over $\mathbb{Q}$ (either definite or indefinite) with discriminant N^- dividing N. We fix isomorphisms $\Phi_q : B \otimes \mathbb{Q}_q \simeq M_2(\mathbb{Q}_q)$ for $q \nmid N^-$. If B is unramified at infinity, we also fix an isomorphism Φ_∞ : $B \otimes \mathbb{R} \simeq M_2(\mathbb{R})$.

In the case when B is ramified at infinity, we first pick a model for B, $B = \mathbb{Q} + \mathbb{Q}a + \mathbb{Q}b + \mathbb{Q}ab$, with $a^2 = -N^-$, $b^2 = -l$ and $ab = -ba$, where l is an auxiliary prime chosen such that

$$\left(\frac{-l}{q}\right) = -1 \text{ if } q \mid N^- \text{ and } q \text{ is odd,}$$
$$l \equiv 3 (\text{mod } 8).$$

Denote by $\mathcal{O}'$ the maximal order in B given by

$$\mathcal{O}' = \mathbb{Z} + \mathbb{Z}\frac{1+b}{2} + \mathbb{Z}\frac{a(1+b)}{2} + \mathbb{Z}\frac{(r+a)b}{l}$$

where r is any integer satisfying $r^2 + N^- \equiv 0 \mod l$. We may assume that the isomorphisms Φ_q are chosen such that $\Phi_q(\mathcal{O}') = M_2(\mathbb{Z}_q)$ for $q \nmid N^-$.

Let $\mathbb{H}$ be the division algebra of Hamilton quaternions, i.e., $\mathbb{H} = \mathbb{R} + \mathbb{R}i + \mathbb{R}j + \mathbb{R}k$ with the relations $i^2 = j^2 = k^2 = -1, ij = -ji = k, jk = -kj = i, ki = -ik = j$ and fix an isomorphism $\Phi_\infty : B \otimes \mathbb{R} \to \mathbb{H}$ characterized by

$$\Phi_\infty : a \mapsto \sqrt{N^-}j, b \mapsto \sqrt{l}i.$$

Note that we can identify the subalgebra of elements of the form $a + bi$ in $\mathbb{H}$ with the field $\mathbb{C}$ of complex numbers and $\mathbb{H} = \mathbb{C} + \mathbb{C}j$. We fix an isomorphism $\rho : \mathbb{H} \otimes_{\mathbb{R}} \mathbb{C} \simeq M_2(\mathbb{C})$ characterized by

$$\rho(\gamma + \delta j) = \begin{pmatrix} \gamma & \delta \\ -\overline{\delta} & \overline{\gamma} \end{pmatrix}$$

for $\gamma, \delta \in \mathbb{C}$. We denote by the same symbol ρ the composite map $(\rho \otimes 1) \circ \Phi_\infty : B_\infty^\times \to \mathrm{GL}_2(\mathbb{C})$. Let F be the subfield of $\mathbb{C}$ generated by $\sqrt{N^-}$ and $\sqrt{-l}$ and $R_0 = \mathcal{O}_{F,(l)}$ the subring of F obtained from $\mathcal{O}_F$ by inverting l. Then one checks immediately that

$$\rho(\mathcal{O}') \subset M_2(R_0).$$

We consider B as a quadratic space over $\mathbb{Q}$, the quadratic form being the reduced norm. Let $\mathrm{GO}(B)$ denote the corresponding orthogonal similitude group. One has a surjection $\kappa : B^\times \times B^\times \to \mathrm{GO}(B)^0$ onto the connected component of $\mathrm{GO}(B)$, given by $(\gamma_1, \gamma_2) \mapsto (x \rightsquigarrow \gamma_1 x \gamma_2^{-1})$, the kernel being a copy of $\mathbb{G}_m$ embedded diagonally. Then there are theta lifts

$$\Theta(\cdot, \psi) : \mathcal{A}_0(G) \to \mathcal{A}_0(G')$$
$$\Theta^t(\cdot, \psi) : \mathcal{A}_0(G') \to \mathcal{A}_0(G)$$

for $G = \mathrm{GL}_2, G' = \mathrm{GO}(B)^0$ (see [6], [7], and [18] for more details). Note that via κ, automorphic representations of G' can be identified with pairs (π_1, π_2), the π_i being representations of $B^\times$ such that $\xi_{\pi_1} \cdot \xi_{\pi_2} = 1$. Here ξ_{π_i} denotes the central character of π_i.

Let π_B denote an automorphic cuspidal representation of $B^\times$, $\overline{\pi}_B$ its complex conjugate and set $\pi = \mathrm{JL}(\pi_B)$.

Theorem 3.1 (Shimizu).

1. $\Theta(\pi, \psi) = \pi_B \times \overline{\pi}_B$.
2. $\Theta^t(\pi_B \times \overline{\pi}_B, \psi) = \pi$.

Suppose now that π corresponds to a holomorphic newform f of weight $2k$ on $\Gamma_0(N)$ with N square-free. We assume that the first Fourier coefficient of f is equal to 1 and denote by the same symbol f the corresponding adelic automorphic form. On $B^\times$, there is no theory of q-expansions and it is not clear how one might pick a canonical element of π_B analogous to the element f in π. However, the situation can be partially remedied as follows. The representation π_B is a restricted tensor product $\pi_B \simeq \otimes_v \pi_{B,v}$ of local representations. For finite primes $v = q$ such that B is split at q, let g_q be a local new vector in $\pi_{B,q}$ as given by Casselman's theorem, i.e., g_q is nonzero and invariant under the action of

$$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}_q), c \in N\mathbb{Z}_q \right\}.$$

Here we have identified $(B \otimes \mathbb{Q}_q)^\times$ with $\mathrm{GL}_2(\mathbb{Q}_q)$ via the isomorphism Φ_q. For finite primes v such that B is ramified at v, the local representation $\pi_{B,v}$ is one-dimensional since π_v is a special representation. In this case, we pick g_v to be any nonzero vector in $\pi_{B,v}$. Finally for $v = \infty$, there are two cases since B is split or ramified at infinity. In the former case, we pick g_∞ to be the unique nonzero vector up to scaling on which $\kappa_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ acts by $e^{2ik\theta}$. In the latter case, one has that the representation $\pi_{B,\infty}$ is isomorphic to

$$\rho_k : B_\infty^\times \to \mathrm{GL}_{2k-1}(\mathbb{C}), \qquad \rho_k = \mathrm{Sym}^{2k-2} \rho \otimes (\det \rho)^{1-k}.$$

Let $V_1 = \mathbb{C}^2$ be the representation space associated to ρ and denote by e_1, e_2 the standard basis. Then the set of vectors $e_1^{\otimes r} \otimes e_2^{\otimes 2k-2-r}, 0 \le r \le 2k-2$ is a basis for V_k, the representation space of ρ_k. Fixing an isomorphism between $\pi_{B,\infty}$ and V_k, we pick $g_\infty^l = e_1^{\otimes r} \otimes e_2^{\otimes 2k-2-r}$. Notice that g_∞^r spans the unique line in $\pi_{B,\infty}$ on which $e^{i\theta} \in \mathbb{C}^{(1)}$ acts by $e^{2i(r-(k-1))\theta}$. Thus if B is indefinite, $g_B = \otimes_v g_v$ in π_B is well-defined up to scaling, while if B is definite, the vector of forms $[g_B^r]$, with $g_B^r = \otimes_{v<\infty} g_v \otimes g_\infty^r$ in π_B is well-defined up to scaling. We will see below that for a given prime p, we can pick g (resp. $[g_B^r]$) in such a way that it is well-defined up to a p-adic unit in K_f.

We will now pick a Schwartz function φ (resp. functions φ^r) in $\mathcal{S}(B_\mathbb{A})$ such that $\theta(f, \varphi, \psi) = \beta g_B$ (resp. $[\theta(f, \varphi^l, \psi)] = \beta[g_B^r]$) in the indefinite (resp. definite) case for some scalar β. Suppose that $N = N^+N^-$ and let $\mathcal{O}$ be the unique Eichler order of level N^+ in B such that

$$\Phi_q(\mathcal{O} \otimes \mathbb{Z}_q) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}_q), c \in N^+\mathbb{Z}_q \right\} \text{ for } q \nmid N^- .$$

Note that for $q \mid N^-$, $\mathcal{O} \otimes \mathbb{Z}_q$ is just the unique maximal order in $B \otimes \mathbb{Q}_q$. Now for $v = q$ a finite prime, set $\varphi_q =$ the charactersitic function of $\mathcal{O} \otimes \mathbb{Z}_q$. If $v = \infty$ and B is indefinite, set

$$\varphi_\infty(\beta) = \frac{1}{\pi} Y(\beta)^k e^{-2\pi(|X(\beta)|^2 + |Y(\beta)|^2)}$$

where for $\beta = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{R})$, $X(\beta) = \frac{1}{2}(a+d) + \frac{i}{2}(b-c)$ and $Y(\beta) = \frac{1}{2}(a-d) + \frac{i}{2}(b+c)$. As usual we have identified $B \otimes \mathbb{R}$ in this case with $M_2(\mathbb{R})$ via Φ_∞. If B is definite, Φ_∞ identifies $B \otimes \mathbb{R}$ with the space of Hamilton quaternions. Set

$$\begin{aligned} \varphi_\infty^r(u + vj) &= \bar{v}^{2l} p_{k-1-|l|}(|u|^2) e^{-2\pi(|u|^2+|v|^2)}, \text{ if } l \geq 0 \\ &= v^{2|l|} p_{k-1-|l|}(|u|^2) e^{-2\pi(|u|^2+|v|^2)}, \text{ if } l \leq 0 \end{aligned}$$

where $l = k - 1 - r$ and p_m is the Laguerre polynomial of degree m, given by

$$p_m(t) = \sum_{j=0}^{m} \binom{l}{j} \frac{(-t)^j}{j!}.$$

Finally, set $\varphi^r = \otimes_v \varphi_v \otimes \varphi^r_\infty$. The following proposition follows easily from computations in [32] and [34].

Proposition 3.2. *Suppose that B is indefinite (resp. definite.) Let $\delta := \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \in B^\times_\infty$ (resp. $\delta := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in B^\times_\infty$.) Then*

$$\theta(f, \varphi, \psi)(x \cdot \delta) = \overline{\theta(f, \varphi, \psi)(x)}.$$

Further, there exist nonzero scalars α, β (resp. α_r, β) such that

(a) $\theta(f, \varphi, \psi) = \beta \cdot (g_B \times g_B)$ (resp. $[\theta(f, \varphi^r, \psi)] = \beta \cdot [g^r_B \times g^r_B]$).
(b) $\theta^t(g_B \times g_B, \varphi, \psi) = \alpha f$ (resp. $\theta^t(g^r_B \times g^r_B, \varphi^r, \psi) = \alpha_r f$).

Note that by our assumption that f occurs on $\Gamma_0(N)$, π and π_B both have trivial central character, hence $\overline{\pi}_B = \pi_B$. If B is indefinite (resp. definite) let Ψ (resp. $\tilde{\Psi}$) be such that $\theta(f, \varphi, \psi) = \Psi \times \Psi$ (resp. $[\theta(f, \varphi^r, \psi)] = \tilde{\Psi} \times \tilde{\Psi}$.)

In the following discussion we write $\theta(f)$ instead of $\theta(f, \varphi, \psi)$ for simplicity of notation. There are two important formulas that are very useful in this situation, namely see-saw duality ([13] and [7]) and the Rallis inner product formula. In the indefinite case, applying see-saw duality gives

$$\begin{aligned} \langle \theta(f), g_B \times g_B \rangle &= \langle f, \theta^t(g_B \times g_B) \rangle \\ \beta \langle g_B, g_B \rangle^2 &= \overline{\alpha} \langle f, f \rangle \end{aligned} \tag{3.1}$$

where $\langle\ ,\ \rangle$ is the Petersson inner product. Next the Rallis inner product formula gives

$$\begin{aligned}\langle\theta(f),\theta(f)\rangle &\sim L(1,ad^0(\pi))\langle f,f\rangle\\ \beta\overline{\beta}\langle g_B,g_B\rangle^2 &\sim \langle f,f\rangle^2\end{aligned}\tag{3.2}$$

Combining (3.1) and (3.2) yields

$$\overline{\alpha}\overline{\beta}\sim\langle f,f\rangle \tag{3.3}$$

and

$$\alpha\overline{\alpha}\sim\langle g_B,g_B\rangle^2. \tag{3.4}$$

Clearly, exactly the same formulas hold also in the definite case, with α,β,g_B being replaced by α_r,β,g_B^r respectively. In particular from (3.3) we see that $\overline{\alpha_r}\sim\overline{\alpha_{r'}}$ and hence $\alpha_r\sim\alpha_{r'}$. This implies also that $\langle g_B^r,g_B^r\rangle\sim\langle g_B^{r'},g_B^{r'}\rangle$.

The indefinite case

In this section we suppose that B is indefinite. The form g_B that we picked in the previous section corresponds in the usual way to a classical modular form on the upper half plane $\mathfrak{H}$ (which we denote simply by the symbol g) with respect to the group $\mathcal{O}^{(1)}$ consisting of the elements of $\mathcal{O}$ with reduced norm. Further we may view $\varsigma=g(z)(2\pi i\cdot dz)^{\otimes k}$ as being a section of the line bundle Ω^k on the curve $X=\mathfrak{H}/\mathcal{O}^{(1)}$. One knows from the work of Shimura that the curve X admits a canonical model $X_{\mathbb{Q}}$ over $\mathbb{Q}$. Let $\mathcal{X}$ denote the minimal regular model of X over $\mathcal{O}_{K_f}$ and denote by ω the relative dualizing sheaf on $\mathcal{X}/\operatorname{spec}\mathcal{O}_{K_f}$. Since the Hecke eigenvalues of g lie in K_f, we may choose g such that $\varsigma\in H^0(X_{K_f},\Omega^k)$ and ς is a p-unit in $H^0(\mathcal{X},\omega^k)$. Thus g is well-defined up to a p-unit in K_f. Fixing such a choice of g, one has:

Theorem 3.3 (Harris–Kudla [6]). *$\beta\in K_f$. Consequently $\langle f,f\rangle/\langle g,g\rangle\in K_f$.*

Indeed, since K_f is totally real, one gets from (3.2) that $\beta\cdot\langle g,g\rangle\sim\langle f,f\rangle$ so that

$$\beta\sim\frac{\langle f,f\rangle}{\langle g,g\rangle}.$$

We are now in a situation where Questions A and B of the introduction make sense, namely, we can ask for information about $v_\lambda(\beta)$ for λ a prime in K_f above p. The answer is provided by the following theorem and corollary which constitute the main results of [18].

Theorem 3.4. *(a)*

$$v_\lambda(\beta) = \min_{K,\chi} v_\lambda(\delta(\pi, K, \chi))$$

where

$$\delta(\pi, K, \chi) := \frac{L(\frac{1}{2}, \pi_K \otimes \chi)}{\Omega_K^{4k}}.$$

Here K ranges over imaginary quadratic fields that are split at N^+ and inert at N^-, χ ranges over unramified Hecke characters of K of type $(k, -k)$ at infinity, and Ω_K is a suitable CM period, i.e., a period of a Neron differential on an elliptic curve that has CM by $\mathcal{O}_K$.

(b)

$$v_\lambda(\delta(\pi, K, \chi)) \geq 0$$

for all K, χ as above. Further if there exists a newform f' of level M dividing N but not divisible by N^- such that $\rho_f \equiv \rho_{f'} \mod \lambda$, then $v_\lambda(\delta(\pi, K, \chi)) > 0$.

(c) $v_\lambda(\beta) \geq 0$. Further, if there exists a newform f' of level M dividing N but not divisible by N^- such that $\rho_f \equiv \rho_{f'} \mod \lambda$, then $v_\lambda(\beta) > 0$.

The reader will note that part (c) of the theorem follows immediately from parts (a) and (b). We first indicate briefly some of the ingredients in the proof of part (a). For K, χ as above, pick a *Heegner* embedding $K \hookrightarrow B$ and set

$$L_\chi(g_B) = j(\alpha, i)^{2k} \int_{K^\times \backslash K_{\mathbb{A}}^\times / K_\infty^\times} g_B(x\alpha)\chi(x) d^\times x$$

for any $\alpha \in \mathrm{SL}_2(\mathbb{R})$ such that $\alpha^{-1} \cdot (K \otimes \mathbb{R}) \cdot \alpha = \left\{ \begin{pmatrix} a & -b \\ b & a \end{pmatrix}, a, b \in \mathbb{R} \right\}$. (We note that such Heegner embeddings exist, if and only if, K is split at the primes dividing N^+ and inert at the primes dividing N^-.) For a suitable choice of measure on $K_{\mathbb{A}}^\times$, the integral above can be interpreted as a sum of values of g at certain CM points associated to K, twisted by the values of χ, and divided by the class number h_K. Now, a λ-integral modular form must take λ-integral values at CM points up to suitable CM periods. Conversely, any form that takes on λ-integral values for a large set of CM points must be λ-integral. Hence one can show roughly that

$$\min_{K,\chi,p \nmid h_K} v_\lambda \left(\frac{L_\chi(g_B)}{\Omega_K^{2k}} \right) = 0. \tag{3.5}$$

On the other hand, by the methods of Waldspurger one can show that

$$\beta L_\chi(g_B)^2 = L_{\chi \times \chi}(\theta(f)) \sim \frac{1}{h_K^2} L(\frac{1}{2}, \pi_K \otimes \chi) \tag{3.6}$$

Part (a) of the theorem follows now by combining (3.5) and (3.6).

Next, we give a brief outline of the proof of part (b). We first assume that p is split in K, and $p \nmid h_K$. By the Rankin–Selberg method

$$L(\frac{1}{2}, \pi_K \otimes \chi) = \langle fE, \theta_\chi \rangle$$

where E is a certain weight-1 Eisenstein series and $\theta_\chi \in S_{2k+1}(\Gamma_1(d_K))$ is the theta function associated to χ. The form fE has integral Fourier coefficients, since E does. Let us expand

$$fE = \gamma \cdot \theta_\chi + H$$

where H is a linear combination of forms orthogonal to θ_χ. Then

$$\frac{\langle fE, \theta_\chi \rangle}{\Omega_K^{4k}} = \gamma \frac{\langle \theta_\chi, \theta_\chi \rangle}{\Omega_K^{4k}} \sim \gamma \cdot \frac{L(\chi(\chi^\rho)^{-1}, 1)}{\Omega_K^{4k}}$$

where $\chi^\rho = \chi \circ \rho$ is the twist of χ by complex conjugation ρ. From Shimura, one knows that both γ and $L(\chi) := L(\chi(\chi^\rho)^{-1}, 1)/\Omega_K^{4k}$ are algebraic; in fact one even knows, for example from the results in [22] that $L(\chi)$ is λ-integral. The problem is that γ is unlikely to be λ-integral. However if γ had a denominator, by multiplying (3.7) by an appropriate power of λ, we would get congruences modulo λ between θ_χ and other forms orthogonal to θ_χ. Let us assume for the moment that $\theta_\chi \equiv h \mod \lambda$ for some eigenform h that is not a theta lift from K. On the one hand, the λ-adic representation $\rho_{h,\lambda}$ associated to h is irreducible even when restricted to $\mathrm{Gal}(\overline{K}/K)$ since h is not a theta lift from K; on the other hand, $\overline{\rho}_{h,\lambda}|_{\mathrm{Gal}(\overline{K}/K)}$ is reducible, being isomorphic mod λ to $\chi_\lambda \oplus \chi_\lambda^\rho = \rho_{\theta_\chi,\lambda}|_{\mathrm{Gal}(\bar{K}/K)}$, where $\chi_\lambda, \chi_\lambda^\rho$ denote the λ-adic characters associated to χ, χ^ρ, respectively. This latter fact can be used to construct a lattice in the representation space of $\rho_{h,\lambda}$ whose reduction modulo λ is an extension of χ_λ^ρ by χ_λ. For simplicity, let us also say that the class number of K is one. If K_∞ is the unique $\mathbb{Z}_p^2$ extension of K, the splitting field of the extension obtained above is an abelian p-extension K' of K_∞ with controlled ramification such that the conjugation action of $\mathrm{Gal}(\overline{K}/K)$ on $\mathrm{Gal}(K'/K_\infty)$ is via the character $\chi_\lambda(\chi_\lambda^\rho)^{-1}$.

The idea that one can construct extensions by the method above is originally due to Ribet [19]; however we need to employ the more refined methods of Wiles [33]. The upshot is that we can construct an extension K'/K_∞, as above, whose size is at least as large as the denominator of γ. In contrast, the main conjecture of Iwasawa theory for K (a theorem of Rubin [21]) can be used to bound the size of such an extension from above by the L-value $L(\chi)$. Thus any possible denominators in γ are cancelled by the numerator of $L(\chi)$ and the product $\gamma \cdot L(\chi)$ is λ-integral as desired. We refer the reader to Chapter 4 of [18] for more details on the above constructions and to Section 5.3 of the same article for the proof that the L-values above are divisible by the expected level-lowering congruence primes.

Remark 3.5. The proof outlined above works whenever p is split in K and $p \nmid h_K$. This is enough to conclude part (c) of the theorem, since one has infinitely many CM points satisfying these conditions. But by parts (a) and (c), we see that part (b) must remain true even is p is inert in K or $p \mid h_K$ (or both.) Thus the *ordinary* case of part (b) is used to prove the *supersingular* case of the same.

Remark 3.6. In fact, one does not need the full main conjecture to deduce the above integrality result but only the *anticyclotomic* part. It is also sometimes possible to prove directly the integrality of $\langle G, \theta_\chi \rangle / \Omega_K^{4k}$ for G in integral form and then deduce the anticyclotomic main conjecture as a consequence. Indeed, this is the approach taken by Tilouine [28] and Hida [12]. The latter article deals also with the case of CM fields. However, the results of these articles require extra hypotheses that may not always be satisfied in our situation. Thus it seems to the author that only the approach above — using the main conjecture rather than deducing it as a consequence — provides the requisite precision needed in our analysis.

Remark 3.7. One would certainly expect conversely, that if $v_\lambda(\langle f, f \rangle / \langle g, g \rangle) > 0$, then λ is a level-lowering congruence prime of the expected type. One might expect an even stronger statement to be true, namely a canonical factorization of this ratio as a product of integers, the terms of the product being indexed by the primes dividing N^- and admitting a geometric interpretation, as is the case for elliptic curves (see Example 2.3). Unfortunately, we have nothing to say about this problem at present for forms of higher weight.

The definite case

In this section, we suppose that B is a definite quaternion algebra. Recall that for every integer r satisfying $0 \leq r \leq (2k-2)$, we have picked a form g_B^r on $B_{\mathbb{A}}^\times$ such that the vector of forms $[g_B^r]$ is well-defined up to a scalar. Set $\tilde{g}_B = [g_B^r]^t$, so that $\tilde{g}_B \in \tilde{S}_k(U)$ where

$$\tilde{S}_k(U) = \{\tilde{g} : B^\times \setminus B_{\mathbb{A}}^\times \to \mathbb{C}^{2k-1} \mid \tilde{g}(x \cdot u u_\infty) = \rho_k(u_\infty)\tilde{g}(x) \ \forall \ u \in U, u_\infty \in B_\infty^\times\}$$

and $U = \prod_q U_q$ is the open compact subgroup of $B_{\mathbb{A}_f}^\times$ given by $U_q = (\mathcal{O} \otimes \mathbb{Z}_q)^\times$.

Since any element of $\tilde{S}_k(U)$ is determined by its values on $B_{\mathbb{A}_f}^\times$, the space $\tilde{S}_k(U)$ is canonically isomorphic to the space $S_k(U)$ given by

$$S_k(U) = \{g : B_{\mathbb{A}_f}^\times / U \to \mathbb{C}^{2k-1} \mid g(\alpha \cdot x) = \rho_k(\alpha^{-1}) g(x) \ \forall \alpha \in B^\times\}.$$

Denote by g_B the element of $S_k(U)$ corresponding to $\tilde{g}_B$. We now follow [27] in defining an integral (or rather p-integral) structure on $S_k(U)$. For R

any ring such that $R_0 \subset R \subset \mathbb{C}$, let $L_k(R)$ be the R-submodule of $\mathbb{C}^{2k-1}$ consisting of vectors all of whose components are in R. The group $B_{\mathbb{A}_f}^\times$ acts on R_0-lattices in $L_k(K)$ via the embedding

$$B_{\mathbb{A}_f}^\times \hookrightarrow (B \otimes \mathbb{A}_{K,f})^\times \xrightarrow{\mu_k \otimes 1} \mathrm{GL}_{2k-1}(\mathbb{A}_{K,f}).$$

Set $L_k(R) \cdot x := L_k(\mathcal{O}_K) \cdot x \otimes R$. We then define $S_k(U; R)$ to be the set of $h \in S_k(U)$ such that $h(x) \in L_k(R) \cdot x^{-1}$ for all $x \in B_{\mathbb{A}_f}^\times$.

Let $K_{0,f} = K_f F$ be the compositum of K_f and F. We may then normalize g_B by requiring that it be a p-unit in $S_k(U, R_p)$ where R_p is the subring of p-integral elements in $K_{0,f}$. With this normalization, it makes sense to study the arithmetic properties of α_r and β. Note that this case is very different from the indefinite case in that $\alpha_r \in \overline{\mathbb{Q}}$ while $\beta / u_+(f)u_-(f) \in \overline{\mathbb{Q}}$.

Let $(,)$ denote the inner product on $S_k(U)$ defined in [27] (and denoted by $\langle , \rangle$ in that article). For $g_B \in S_k(U)$ and $\delta = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $g_B' := \rho_k(\delta)\overline{g} \in S_k(U)$ and it is easy to see that

$$\langle \tilde{g}_B, \tilde{g}_B \rangle = \sum_r \langle g_B^r, g_B^r \rangle = (g_B, g_B').$$

Note that $\langle f, f \rangle^2 \sim \beta\overline{\beta} \langle \tilde{g}_B, \tilde{g}_B \rangle^2 = \langle \tilde{\Psi}, \tilde{\Psi} \rangle^2 = (\Psi, \Psi)^2 = \beta^2 (g_B, g_B)^2$ since $\overline{\Psi(x) \cdot \delta} = \Psi(x)$ (and hence $\Psi' = \Psi$).

Set $\delta_g = (g_B, g_B)$. As in the indefinite case, one may define an invariant η_g that counts congruences satisfied by g; one always has that $(\eta_g) \subseteq \delta_g$ and in good situations (namely when some freeness condition holds) one has also $(\delta_g) = \eta_g$. Now

$$\beta \sim \frac{\langle f, f \rangle}{\delta_g} \sim \frac{\delta_f}{\delta_g} u_+(f) u_-(f)$$

and $\alpha \sim (g_B, g_B)$. Since $\eta_g \subseteq \eta_f$, we obtain:

Theorem 3.8. *(a)* $v_\lambda(\alpha) \geq 0$.
(b) $v_\lambda\left(\frac{\beta}{u_+(f)u_-(f)}\right) = v_\lambda\left(\frac{\delta_f}{\delta_g}\right)$. *In particular, if* $(\delta_f) = \eta_f$,

$$v_\lambda\left(\frac{\beta}{u_+(f)u_-(f)}\right) \geq 0.$$

We now explain the relation between this result and Rankin–Selberg L-values. Let K be an imaginary quadratic field that is split at N^+ and inert at N^-, $i : K \hookrightarrow B$ is a Heegner embedding with $p \nmid h_K$. For any integer r with $0 \leq r \leq 2k-2$, let χ_r be an unramified Hecke character of K of type $(r_0, -r_0)$ at infinity, where $r_0 = r - (k-1)$. With a suitable choice of measure, one defines

$$L_{\chi_r}(g_B^r) = \int_{K^\times \backslash K_{\mathbb{A}}^\times / K_\infty^\times} g_B^r(x\gamma)\chi(x)d^\times x$$

for any $\gamma \in (B \otimes \mathbb{R})^\times = \mathbb{H}^\times$ such that $\gamma^{-1}(K \otimes \mathbb{R})\gamma = \mathbb{C} \subset \mathbb{H}$. Again, by methods of Waldspurger one can prove that

$$\beta L_{\chi_r}(g_B^r)^2 = L_{\chi_r \times \chi_r}(\theta(f, \varphi_r)) \sim L(\frac{1}{2}, \pi_K \otimes \chi_r).$$

Combining this with (3.2) yields

$$|L_{\chi_r}(g_B^r)|^2 \sim L(\frac{1}{2}, \pi_K \otimes \chi_r)\frac{\langle g_B^r, g_B^r\rangle}{\langle f, f\rangle}$$

which is just one form of Gross's special value formula.

The following integrality criterion for forms on $B^\times$ follows quite easily from the equidistribution theorem (Theorem 10) of [14].

Proposition 3.9 (Integrality criterion for forms on $B^\times$). *A form $\tilde{\Psi}' = [\Psi_r']$ is p-integral if and only if for some Heegner point $K \hookrightarrow B$ with $p \nmid h_K$ and $h_K >> 0$, and all unramified characters χ_r of $K_{\mathbb{A}}^\times$ of infinity-type $(r_0, -r_0)$, $0 \le r \le 2k-2$,*

$$L_{\chi_r}(\Psi_r') := \int_{K^\times K_\infty^\times \backslash K_{\mathbb{A}}^\times} \Psi_r'(x)\chi_r(x)d^\times x \tag{3.7}$$

is p-integral.

Note that the expression (3.7) is a finite sum of the values Ψ_r' twisted by the values of the character χ_r. Applying the criterion to the form $\tilde{\Psi}$ constructed earlier and using Theorem 3.8, we see that

Theorem 3.10. *For K, r, χ_r as above, and λ any prime above p,*

$$v_\lambda\left(\frac{L(\frac{1}{2}, \pi_K \times \chi_r)}{u_+(f)u_-(f)}\right) \ge v_\lambda(\frac{\delta_f}{\delta_g}).$$

Further, for any K with $h_K >> 0$, there exists a pair (r, χ_r) such that equality holds.

3.2 The dual pair $(\widetilde{\mathrm{SL}_2}, \mathrm{O}(V))$, $V = B^0$

The indefinite case

Note that Theorem 3.4 above does not address the periods $u_\pm(f), u_\pm(g)$; rather, it pertains only to the Petersson norms which are products of periods. It turns out to be much harder to prove results about individual periods. By studying a relevant theta correspondence, we are able to prove the following result about ratios of periods. We denote by $A(f,d)$ the algebraic part of the L-value $L(\frac{1}{2},\pi_f\otimes\chi_d)$ for d any fundamental quadratic discriminant and χ_d the corresponding character, i.e., $A(f,d)=\mathfrak{g}(\chi_d)L(\frac{1}{2},\pi_f\otimes\chi_d)/u_\tau(f)$ for $\tau=(-1)^k\mathrm{sign}(d)$. It is known that $A(f,d)\in K_f$ and that it is a p-adic integer at least when p is not an Eisenstein prime for f.

Theorem 3.11. *Suppose that N is odd and square-free.*

(a) Let $\sigma\in Aut(\mathbb{C}/\mathbb{Q})$. Then

$$\left(\frac{u_\pm(f)}{u_\pm(g)}\right)^\sigma=\frac{u_\pm(f^\sigma)}{u_\pm(g^\sigma)}.$$

(b) Suppose there exists a quadratic discriminant d such that $p\nmid A(f,d)$. Then

$$v_\lambda\left(\frac{u_\tau(f)}{u_\tau(g)}\right)\geq 0$$

where $\tau=(-1)^k\mathrm{sign}(d)$.

In the case f has weight 2, one can use the rationality of period ratios provided by part (a) to construct directly isogenies between quotients of $J_0(N)$ and $Jac(X)$, completely independent of Faltings' isogeny theorem. Further in the case when $k=2$, one also has applications relating to questions about p-divisibility and the indivisibility of central values of quadratic twists. (See [15] and [17] for more details on these applications.)

The proofs of the above results are based on studying the p-adic properties of the theta lifting for the dual pair $(\widetilde{\mathrm{SL}_2},\mathrm{O}(V))$ with V the space of trace zero elements in B. The automorphic theory in this case has been worked out in great detail in three beautiful articles of Waldspurger ([29], [30] and [31]). In the arithmetic theory there are three complications that arise. Firstly, there is not one automorphic form on $\widetilde{\mathrm{SL}_2}$ but rather a *packet* of forms that corresponds to g_B. Secondly, there is no good theory of newforms for forms of half-integral weight. Lastly, while one can again measure arithmeticity on $PB^\times=\mathrm{SO}(V)$ by means of period integrals on tori, the relevant period integrals are not related to a Rankin–Selberg L-value as in the case of $\mathrm{O}(B)$. However, for a suitable choice of ψ and $\varphi\in V(\mathbb{A})$ and a suitable form h that has weight $k+\frac{1}{2}$ and that is p-adically normalized, one can show:

Theorem 3.12. *(a) $\theta^t(g,\varphi,\psi)=\alpha u_\pm(g)h$ for some scalar α.*
(b) $\theta(h,\varphi,\psi)=\beta g$ for some scalar β.
(c) $\alpha,\beta\in\overline{\mathbb{Q}}$. Further $v_\lambda(\alpha)\geq 0$ and $v_\lambda(\beta)\geq 0$.

The proof of the above theorem (especially the p-integrality of β) is rather intricate, so we refer the reader to the article [15] for more details.

The definite case

It is not hard to show in this case that for suitable choices of φ, ψ, and h, $\theta^t(g,\varphi,\psi) = \alpha h$ and $\theta(h,\varphi,\psi) = \beta u_\pm(f)g$ for some scalars α,β. Unfortunately, the author does not know how to prove in this case the analog of Theorem 3.8(b), i.e., the p-integrality of β. One would certainly conjecture that:

Conjecture 3.13. $v_\lambda\left(\frac{\beta}{u_\pm(f)}\right) \geq 0.$

However the previous methods of proofs break down; one seems to require rather refined information, about Petersson inner products and congruences of half-integral weight forms, that is not presently available. More precisely, one is lead to conjecture that the algebraic parts of Petersson inner products of half-integral weight forms count congruences satisfied by these forms just as in the integral weight setting. The reader is referred to [17] for a discussion of this issue.

4 A conjecture on Petersson inner products of quaternionic modular forms over totally real fields

In this section, we consider conjectural generalizations to Hilbert modular forms and their quaternionic analogs over totally real fields. Let F be a totally real field, $\Sigma_{F,\infty}$ (resp. $\Sigma_{F,\mathrm{fin}}$, resp. Σ_F) the set of infinite places (resp. finite places, resp. all places) of F. Let $\pi = \otimes_v \pi_v$ the automorphic representation of $\mathrm{GL}_2(\mathbb{A}_F)$ corresponding to a holomorphic Hilbert modular form f with even weights at infinity. We will assume for simplicity that π_v is a special representation for all finite places v of F such that π_v is discrete series. Let B be a quaternion algebra over F such that π admits a Jacquet–Langlands transfer π_B to B, i.e., such that for all places v where B is ramified, π_v is a discrete series representation. As explained before, we can pick a non-zero element $g_B \in \pi_B$ that is well-defined up to multiplication by a scalar. We suppose that g_B is arithmetically normalized and consider the Petersson inner product $\langle g_B, g_B\rangle$.

Remark 4.1. The form g_B corresponds to a section of an *automorphic* vector bundle V on a Shimura variety X_B attached to B. Such vector bundles are known to have canonical models over specified number fields based on the work of Harris [4]. Thus it is perfectly clear how to normalize g_B up to an element in a specific number field. However, to normalize g_B up to a p-adic unit, one needs to construct canonical integral models of X and V over suitable p-adic rings. We will assume in what follows that such models exist, and hence that g_B may be normalized up to a p-adic unit.

The problem of relating the numbers $\langle g_B, g_B\rangle$ as B varies was first considered by Shimura in the early 80's. Shimura proved that up to algebraic factors, this Petersson inner product only depends on the set of infinite places at which B is unramified. Further, he conjectured that to each infinite place v of F, one can associate a (transcendental) number c_v depending only on π, such that

$$\langle g_B, g_B\rangle \sim_{\overline{\mathbb{Q}}^*} \prod_{\substack{v\in\Sigma_{F,\infty}\\ v\nmid \mathrm{disc}B}} c_v. \tag{4.1}$$

This conjecture was proved by Harris ([5]) under the hypothesis that for at least one finite place v, π_v is discrete series. Notice that (4.1) implies that

$$\langle f, f\rangle \sim_{\overline{\mathbb{Q}}^*} \prod_{v\in\Sigma_{F,\infty}} c_v$$

and thus

$$\langle g_B, g_B\rangle \sim_{\overline{\mathbb{Q}}^*} \frac{\langle f, f\rangle}{\prod_{\substack{v\in\Sigma_{F,\infty}\\ v\mid \mathrm{disc}B}} c_v}.$$

On the other hand, by Example 2.3, for forms corresponding to elliptic curves over $\mathbb{Q}$, with B an indefinite quaternion algebra, we have

$$\langle g_B, g_B\rangle \sim \frac{\langle f, f\rangle}{\prod_{\substack{v\in\Sigma_{F,\mathrm{fin}}\\ v\mid \mathrm{disc}B}} c_v}$$

where $\sim$ now denotes equality up to p-units, and the c_v's are orders of certain component groups. This leads us to make the following conjecture in the totally real case.

Conjecture 4.2. *Suppose that p is a generic prime for (F,π), i.e., p is prime to $\mathrm{disc}_{F/\mathbb{Q}}$, $\deg_{F/\mathbb{Q}}$, the class number of F, and the level of π. Then for each place v of F such that π_v is a discrete series, there exists a complex number c_v such that*

$$\langle g_B, g_B\rangle \sim \frac{\langle f, f\rangle}{\prod_{\substack{v\in\Sigma_F\\ v\mid \mathrm{disc}B}} c_v}$$

where $\sim$ denotes equality up to a p-unit.

Remark 4.3. One would expect the c_v's to be transcendental for v infinite and algebraic integers for v finite. Further for finite v, c_v should measure level-lowering congruences at v satisfied by π. Note that the c_v's are not local invariants, i.e., they are not determined by the local representation π_v but rather depend very much on the global representation π.

Remark 4.4. It is well-known that $\langle f, f\rangle \sim L(\mathrm{ad}^0\pi, 1)$ where $\mathrm{ad}^0\pi$ denotes the adjoint representation. It is an interesting problem, suggested to the author by Colmez, to study the relation between the conjecture above and the Bloch–Kato conjecture for the adjoint L-value. We hope to take this up in future work.

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