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On the Fourier coefficients of modular forms of half-integral weight

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Abstract. We prove a formula relating the Fourier coefficients of a modular form of half-integral weight to the special values of L-functions. The form in question is an explicit theta lift from the multiplicative group of an indefinite quaternion algebra over $\mathbb{Q}$. This formula has applications to proving the nonvanishing of this lift and to an explicit version of the Rallis inner product formula.

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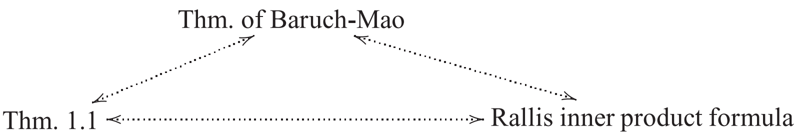
1 Introduction

The origins of this article lie in a remarkable proportionality relation discovered by Waldspurger [12] between the squares of the Fourier coefficients of a half-integral weight modular form h and special values of L-functions attached to the Shimura lift f of h. Recently, by a different method (namely Jacquet's relative trace formula), Baruch and Mao (generalizing results of Kohnen-Zagier [2], Kohnen [1] etc.) have established in some cases a more precise relation between the squares of the *absolute values* of the Fourier coefficients and special values, thus computing the absolute value of the proportionality constant occurring in Waldspurger's work.

The aim of this article is to establish yet another formula (see Thm. 1.1 below) for the squares of the absolute values of Fourier coefficients of a certain explicitly constructed half-integral weight form. This formula has several applications. Firstly, it can be used to show that a certain explicit theta lift, from $PB^\times$ to $\widetilde{\mathrm{SL}_2}$, for B an indefinite quaternion algebra over $\mathbb{Q}$, is nonvanishing. Secondly, combining this formula with the formula of Baruch-Mao mentioned above, one may obtain a completely precise version of the Rallis inner product formula (see [7] Sec. 7.1.) for certain theta lifts from $\widetilde{\mathrm{SL}_2}$ to $PB^\times$. Both Thm. 1.1 and this explicit version of the Rallis inner product formula are crucial ingredients in the forthcoming articles [7], [8], on the arithmetic properties of the theta correspondence and the p-adic properties of central L-values of quadratic twists of elliptic curves. Since the relation between the formulae above may be somewhat confusing, we remark that it is explained by the diagram below, in which any two of the formulae imply the third.



To explain the statement of Thm. 1.1, we need some notation. Let f be a holomorphic modular form of weight $2k$, trivial central character and odd squarefree level N. Let $\boldsymbol{\chi}$ be a finite order character of conductor N' dividing $4N$, and suppose $\tau = 0$ or 1 is such that $\chi = \boldsymbol{\chi} \cdot \left(\frac{-1}{\cdot}\right)^{k+\tau}$ is unramified at 2. Set $\chi_0 = \boldsymbol{\chi} \cdot \left(\frac{-1}{\cdot}\right)^{k}$. Also let w_q be the sign of the Atkin-Lehner involution attached to f at q. Denote by π the automorphic representation of $\mathrm{PGL}_2(\mathbb{A})$ attached to f and let f_χ be a newform in $\pi \otimes \chi$ normalized to have its first Fourier coefficient equal to 1.

Suppose that B is an indefinite quaternion algebra of discriminant $N^- \mid N$ and denote by g_χ a newform in $\pi' \otimes \chi$ where π' is the Jacquet-Langlands lift of π. Let ν be an odd fundamental quadratic discriminant satisfying $\mathrm{sign}(\nu) = (-1)^\tau$ as well as

(1.1) (a) If $q \mid N, q \nmid \nu, \chi_{0,q}(-1) = \alpha_q \chi_{\nu,q}(q) w_q$.

(b) If $q \mid N, q \mid \nu, \chi_{0,q}(-1) = \alpha_q$.

where $\alpha_q = \left(\frac{B}{q}\right)$. In case (b) we also assume for simplicity that $\chi_{0,q}$ is ramified exactly when $q \mid N^-$. Let $V = \{x \in B \mid \mathrm{tr}(x) = 0\}$ and $\varphi \in \mathcal{S}(V(\mathbb{A}))$ the Schwartz function defined in Sec. 3.2 of [7]. Put $s = g_\chi \otimes (\chi^{-1}\chi_\nu) \in \pi' \otimes \chi_\nu$, $\psi' = \psi^{1/|\nu|}$ where ψ is the usual additive character on $\mathbb{Q} \setminus \mathbb{A}_\mathbb{Q}$. Finally, denote by h the theta lift $h := t(\psi', \varphi, s)$ on $\widetilde{\mathrm{SL}_2}(\mathbb{A})$. Then we have

Theorem 1.1. (1) $h \in S_{k+\frac{1}{2}}(M, \chi)$ *where* $M = \mathrm{lcm}(NN', 4)$.

(2) *Let ξ be any positive integer and set $\xi_0 := (-1)^\tau \xi$. Then $a_\xi(h) = 0$ unless the following conditions are satisfied:*

(a) *For all* $q \mid N, q \nmid N'$, $\left(\frac{\xi_0}{q}\right) \neq -w_q$,
(b) *For all* $q \mid N'$, $\left(\frac{\xi_0}{q}\right) = \chi_{0,q}(-1)w_q = \chi_q(-1)w_q$,
(c) $\xi_0 \equiv 0, 1 \mod 4$.

If (a),(b),(c) *are satisfied, and* ξ_0 *is a fundamental quadratic discriminant, then*

$$|a_\xi(h)|^2 = C(f,\chi,\nu)\pi^{-2k}|\nu\xi|^{k-\frac{1}{2}} L\left(\frac{1}{2}, \pi_f \otimes \chi_\nu\right) L\left(\frac{1}{2}, \pi_f \otimes \chi_{\xi_0}\right) \cdot \frac{\langle g_\chi, g_\chi \rangle}{\langle f_\chi, f_\chi \rangle}.$$

where $\langle , \rangle$ *denotes the Petersson inner product and* $C(f,\chi,\nu) = \prod_{q \leq \infty} C_q$, *where* C_q *are the constants listed in Sec. 5. In particular, if* $p > k$ *and* $p \nmid \tilde{N} := \prod_{q|N} q(q^2-1)$, *then* $C(f,\chi,\nu)$ *is a* p*-unit.**

We now explain briefly the idea of the proof. The Fourier coefficient in question is readily expressible as a sum of integrals of g_χ against certain characters of real quadratic tori $K^\times \hookrightarrow B^\times$. One shows first that up to an explicit scalar, it is in fact equal to a single such toric integral. Next one applies a method of Waldspurger (from his article [13]) to show that the absolute value squared of this torus integral is given by the L-value above. This method is now quite familiar and has been exploited in the recent articles [6], [5], [16] to relate period integrals along tori and L-values. The case of real quadratic fields has been worked out in detail in Popa's article [5] with some assumptions on the local ramification. However the particular problem that confronts us is different in two aspects described below, which lends to the novelty of this article.

Firstly, the articles [5], [16] express an L-value as a period integral, for some convenient choice of embedding $j : K^\times \hookrightarrow B^\times$. In our case, it is not the L-value but the period integral that appears in the arithmetic theory of the theta correspondence, and one must prove a formula relating it to the L-value in question. Consequently, one cannot make a convenient choice of embedding j; instead we need to deal systematically with all the different possibilities for j that occur naturally in the formulae for the Fourier coefficients of the theta lift. This leads to the second main difference, namely that we need to consider several more possibilities as regards the local ramification in the ramified zeta integrals than are dealt with in the articles cited above.

We remark that it would be very useful to have a general formula for B a quaternion algebra over an arbitrary number field K, relating the period integral of a newform on B to an L-value for *any* choice of $j : K \hookrightarrow B$ (K quadratic over F) and for *all* possible choices of local ramification. By Waldspurger's formula this is a purely local issue involving computations of ramified zeta integrals. Yet, in that generality, it would be significantly more complicated than the few cases considered in this article.

The present article is organized as follows. In Section 2.1, we set up the main theme of this article, namely a certain explicit theta lift from $PB^\times$ to $\widetilde{\mathrm{SL}_2}$, and in Section 2.2, derive a formula relating the Fourier coefficients of this lift to certain integrals along tori. Next, we recall in Section 3 the method of Waldspurger to compute the squares of absolute values of such period integrals using the dual pair $(\mathrm{GL}_2, \mathrm{GO}(B))$. This method expresses these

* In fact, with a little more care, our proof can easily be modified to prove a formula for the squares of the Fourier coefficients and not just their absolute values.

quantities as the value at $1/2$ of a certain global zeta integral. In Sec. 4 we compute the local zeta integrals in all the ramified cases that have not appeared in previous articles on this subject. The results are put together in Sec. 5 to obtain the final formula of Thm. 1.1.

2 Fourier coefficients and toric integrals

2.1 Ternary forms: the dual pair $(\widetilde{\mathrm{SL}_2}, PB^\times)$ and the basic setup

The reader is referred to the introduction and [7] §2.1 for the notation used below. We recall first some facts about the Weil representation. Let W be a two dimensional symplectic vector space. The metaplectic cover $\widetilde{\mathrm{Sp}}(W \otimes V)$ splits over the orthogonal group $\mathrm{O}(V)$ which is identified with $PB^\times \times \langle c \rangle$, where $\langle c \rangle$ is the cyclic group of order 2 generated by the main involution and the action of $\beta \in PB^\times$ on V is given by $\beta \cdot (x) = \beta x \beta^{-1}$. Let $\widetilde{\mathrm{S}}_v$ (resp. $\widetilde{\mathrm{S}}_{\mathbb{A}}$) denote the twofold metaplectic cover of $\mathrm{SL}_2(\mathbb{Q}_v)$ (resp. $\mathrm{SL}_2(\mathbb{A})$.) Then for each place v of $\mathbb{Q}$, the Weil representation of $\widetilde{\mathrm{Sp}}(W \otimes V)_v$ yields by restriction a representation of $\widetilde{\mathrm{S}}_v \times PB_v^\times$ on $\mathcal{S}_{\psi'}(V \otimes \mathbb{Q}_v)$ denoted $\omega_{\psi'}$. The restriction of $\omega_{\psi'}$ to $\widetilde{\mathrm{S}}_v$ is a genuine representation of $\widetilde{\mathrm{S}}_v$, denoted $r_{\psi'}$, satisfying

$$(2.1) \qquad r_{\psi'}(\mathbf{n})\varphi(x) = \psi'(nQ(x))\varphi(x)$$

$$(2.2) \qquad r_{\psi'}(\mathbf{d}(a))\varphi(x) = \mu_{\psi'}(a)(a,-1)_v|a|^{3/2}\varphi(ax)$$

$$(2.3) \qquad r_{\psi'}(w,\epsilon)\varphi(x) = \epsilon\gamma_{\psi',Q}\mathcal{F}_{\psi'}(\varphi)$$

where we write ψ' instead of ψ'_v. The Haar measure on $V \otimes \mathbb{Q}_v$ is picked to be autodual with respect to the pairing $(x_1,x_2) \mapsto \psi'(\langle x_1,x_2 \rangle)$. In future we will write $\mathcal{F}(\varphi)$ for $\mathcal{F}_{\psi'}(\varphi)$. Thus $\omega_{\psi'}(\sigma,\beta) = r_{\psi'}(\sigma)R(\beta)$ where $R(\beta)\varphi(v) = \varphi(\beta^{-1}v\beta)$.

Let $\mathcal{A}'_0$ and $\tilde{\mathcal{A}}_0$ denote the spaces of cusp forms on $PB^\times(\mathbb{A})$ and $\widetilde{\mathrm{S}}_{\mathbb{A}}$ respectively. For $s \in \mathcal{A}'_0$, $\varphi \in \mathcal{S}_{\psi'}(V_{\mathbb{A}})$, define

$$\theta(\psi',\varphi,\sigma,\beta) = \sum_{x \in V} r_{\psi'}(\sigma)R(\beta)\varphi(x)$$

$$t_{\psi'}(\varphi,\sigma,s) = \int_{PB^\times \backslash PB^\times_{\mathbb{A}}} \theta(\psi',\varphi,\sigma,\beta)s(\beta)d^\times\beta$$

where the measure is the usual Tamagawa measure. We now choose s as in the introduction and $\varphi = \prod_v \varphi_v$ to be the Schwartz function defined in Sec. 3.2 of [7]. It is shown then in [7], Prop. 3.4 that $t_{\psi'}(\varphi,\sigma,s) = t_h$ for a unique $h \in S_{k+\frac{1}{2}}(M,\chi)$.

We recall now some additional notation defined in [7] that we will need. Let $\mathcal{O}_N$ be an Eichler order of level N^+ in B. Let $\mathcal{O}'(\chi)$ be the unique Eichler order in B such that $\mathcal{O}'(\chi) \otimes \mathbb{Z}_q = \mathcal{O}_N \otimes \mathbb{Z}_q$, unless $q \mid N'$ and $q \mid N^+$, in which case

$$\Phi_q(\mathcal{O}'(\chi) \otimes \mathbb{Z}_q) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}_q), c \equiv 0 \mod q^2 \right\},$$

and define open compact subgroups $U_0, U_0(\chi) \subseteq B^\times_{\mathbb{A}_f}$ by
(1) $U_0 = \prod_q U_{0,q}$ where $U_{0,q} = (\mathbb{O}_N \otimes \mathbb{Z}_q)^\times$.
(2) $U_0(\chi) = \prod_q U_{0,q}(\chi)$, where $U_{0,q}(\chi) = (\mathbb{O}'(\chi) \otimes \mathbb{Z}_q)^\times$.

We define below a character $\tilde{\omega}_\chi$ on $U_{0,q}(\chi)$ such that $\tilde{\omega}_\chi|_{\hat{\mathbb{Z}}^\times} = \chi^2$. Firstly, for each q define $\tilde{\omega}_{\chi,q}$ on $U_{0,q}(\chi)$ as follows:

- For $q \nmid N_\chi$, $\tilde{\omega}_{\chi,q}(u) = 1$.
- For $q \mid N_\chi$ and $q \mid N^+$, $\tilde{\omega}_{\chi,q}(u) = \chi_q(d)^2$ for $u = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.
- For $q \mid N_\chi$ and $q \mid N^-$, $\tilde{\omega}_{\chi,q}(u) = \chi_q(\mathrm{Nm}(u))$.

Then, set $\tilde{\omega}_\chi = \prod_q \tilde{\omega}_{\chi,q}$ on $U_{0,q}(\chi)$. Now let Γ (resp. Γ_χ) be the group of norm 1 units in $\mathbb{O}_N$ (resp. $\mathbb{O}'(\chi)$) and define χ' to be the restriction of $\tilde{\omega}_\chi^{-1}$ to $\Gamma_\chi \subseteq U_{0,q}$.

2.2 Integrals along tori and Fourier coefficients of the theta lift

We would like to write down a formula that expresses the Fourier coefficients of h as a sum of period integrals. It is convenient to begin with some generalities. Let $\mathbb{O}'$ be an order in B, U' the open compact subgroup of $B^\times_{\mathbb{A}_f}$ given by $U' = \prod_q U'_q$, $U'_q = (\mathbb{O}' \otimes \mathbb{Z}_q)^\times$. Let $\Gamma' = B^\times \cap (U'(B^\times_\infty)^+)$, $\tilde{\omega}$ be a character of U' and set $\omega' = \tilde{\omega}^{-1}|_{\Gamma'}$. Suppose $g' \in S_{2k}(\Gamma', \omega')$ and $s_{g'} \in S_{2k}(U', \tilde{\omega})$ is the corresponding adelic modular form. Let ω be the unique finite order character of $\mathbb{Q}^\times_{\mathbb{A}}$ such that $\tilde{\omega}|_{U' \cap \mathbb{Q}^\times_{\mathbb{A}_f}} = \omega|_{U' \cap \mathbb{Q}^\times_{\mathbb{A}_f}}$. Thus the central character of $s_{g'}$ is ω.

For $w \in \mathbb{C}$ and $\alpha \in V \otimes_{\mathbb{R}} \mathbb{C}$, define

$$[\alpha, w] = \begin{pmatrix} w & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \alpha \begin{pmatrix} w \\ 1 \end{pmatrix} = cw^2 - 2bw + a$$

$$\text{if } \alpha = \begin{pmatrix} b & -a \\ c & -b \end{pmatrix}.$$

For $x \in V$, let $G_x = \{h \in \mathrm{SL}_2(\mathbb{R}), h^{-1}xh = x\}$, $\Gamma'_x = G_x \cap \Gamma'$ and suppose that $\omega'|_{\Gamma'_x}$ is the trivial character. Then we may define (as in [9] (2.5); see same reference for normalization of measures)

$$P(g', x, \Gamma') = \int_{\Gamma'_x \backslash G_x} [x, hw]^k g'(hw) d(\Gamma'_x h)$$

for any $w \in \mathfrak{H}$, since the condition $\omega'|_{\Gamma'_x} = 1$ implies that the integrand is Γ'_x invariant. Denote by V^* the set of $x \in V$ such that $\mathrm{Nm}(x) < 0$. By [9] Lemma 2.1, $P(g', x, \Gamma')$ is independent of the choice of w and is equal to 0 unless $x \in V^*$. Let $R(\Gamma')$ be the set of equivalence classes in V^* for the conjugation action of Γ' and for $\mathcal{C} \in R(\Gamma')$, set $Q(\mathcal{C}) = -\mathrm{Nm}(x)$ for any choice of $x \in \mathcal{C}$. By [9] (2.6), $P(g', x, \Gamma')$ only depends on the class of x in $R(\Gamma')$. Thus for $\mathcal{C} \in R(\Gamma')$ we may set $P(g', \mathcal{C}, \Gamma') = P(g', x, \Gamma')$ for any choice of $x \in \mathcal{C}$.

Let $F \subset \mathbb{C}$ be any real quadratic field that embeds in B, $j : F \hookrightarrow B$ an embedding, η a finite order character of $F^\times_{\mathbb{A}}$ such that $\eta|_{\mathbb{Q}^\times_{\mathbb{A}}} = \omega^{-1}$ and suppose that ω_∞ is the trivial character.

(Secretly, $\mathcal{O}' = \mathcal{O}'(\chi)$, $U' = U_0(\chi)$, $g' = g_\chi$, $F = \mathbb{Q}(\sqrt{|\nu|\xi})$, $\eta = (\chi_\nu\chi^{-1}) \circ N_{F/\mathbb{Q}}, \omega' = \chi', \tilde{\omega} = \tilde{\omega}_\chi, \omega = \chi^2$.) Suppose $F = \mathbb{Q}+\mathbb{Q}\alpha$, with $\mathrm{tr}(\alpha) = 0$ and $\alpha > 0$. Pick $\gamma_\infty \in \mathrm{SL}_2(\mathbb{R})$ such that $\gamma_\infty^{-1} \cdot \Phi_\infty(j(\alpha))\gamma_\infty = \begin{pmatrix} \alpha & 0 \\ 0 & -\alpha \end{pmatrix}$, and consider the integral

$$L_{\eta,j}(g') = \int_{F^\times \mathbb{Q}_\mathbb{A}^\times \backslash F_\mathbb{A}^\times} s_{g'}(x\gamma_\infty)\eta(x)d^\times x.$$

The assumption $\eta|_{\mathbb{Q}_\mathbb{A}^\times} = \omega^{-1}$ (resp. that ω_∞ is the trivial character) ensures that the integrand above is $\mathbb{Q}_\mathbb{A}^\times$ invariant (resp. that the integral is independent of the choice of γ_∞.) Here we pick our measure conventions to match those of [5].

Suppose $\mathcal{O}' \cap F = \mathbb{Z}+\mathbb{Z}c\delta$, where $\mathbb{Z}+\mathbb{Z}\delta$ is the maximal order of F. Let $\mathcal{O}_{F,c} = \mathbb{Z}+\mathbb{Z}c\delta$, and $U_{F,c}$ be the open compact in $F_{\mathbb{A}_f}^\times$ given by $U_{F,c} = \hat{\mathcal{O}}_{F,c}^\times$, so that $s_{g'}(x)$ and $\eta(x)$ are invariant under $U_{F,c}$. Let $x_i \in F_{\mathbb{A}_f}^\times$ be such that

$$F_\mathbb{A}^\times = \textstyle\bigsqcup_{i=1}^{h'} F^\times \cdot U_{F,c} \cdot (F_\infty^\times)x_i, \qquad h' = [F_\mathbb{A}^\times : F^\times U_{F,c} F_\infty^\times]$$

and suppose

$$j(x_i) = \gamma_i g_{U',i} = g_i(g_{U',i} \cdot \gamma_i^{-1}), g_i \in B^\times, g_{U',i} \in U', \gamma_i \in (B_\infty^\times)^+, \gamma_i = g_i.$$

Then

$$\begin{aligned}\mathrm{vol}(U_{F,c})^{-1}L_{\eta,j}(g') &= \sum_{i=1}^{h'} \eta(x_i)\tilde{\omega}_\chi(g_{U',i}) \int_{(F^\times \cap U_{F,c})\cdot(\mathbb{Q}_\infty^\times)\backslash(F_\infty^\times)} s_{g'}(\gamma_i^{-1}x\gamma_\infty)d^\times x \\ &= (-2\alpha'\iota)^{-k} \sum_{i=1}^{h'} \eta(x_i)\tilde{\omega}(g_{U',i})P(g', j(\alpha')^{x_i}, \Gamma')\end{aligned}$$

where α' is any nonzero element of F with trace 0, $j(\alpha')^{x_i} = \gamma_i^{-1}j(\alpha')\gamma_i$ and $\iota = \sqrt{-1}$.

We now specialize our discussion to the case of interest, namely $g' = g_\chi$, $F = \mathbb{Q}(\sqrt{|\nu|\xi})$, $\eta = (\chi_\nu\chi^{-1}) \circ N_{F/\mathbb{Q}}, \omega' = \chi', \tilde{\omega} = \tilde{\omega}_\chi, \omega = \chi^2, \Gamma' = \Gamma_\chi$. Note that

$$\begin{aligned}\varphi_{fin}(j(\alpha)^{x_i}) &= \varphi_{fin}(\gamma_i^{-1}j(\alpha)\gamma_i) = \varphi_{fin}(g_{U',i}j(x_i)^{-1}j(\alpha)j(x_i)g_{U',i}^{-1}) \\ &= \varphi_{fin}(g_{U',i}j(\alpha)g_{U',i}^{-1}) = (\tilde{\omega}_\chi \cdot ((\chi_\nu\chi) \circ \mathrm{Nm})^{-1})(g_{U',i})\varphi_{fin}(j(\alpha)) \\ &= \tilde{\omega}_\chi(g_{U',i})\eta(x_i)\varphi_{fin}(j(\alpha))\end{aligned}$$

since $(\chi_\nu\chi^{-1})(\mathrm{Nm}(g_{U',i})) = (\chi_\nu\chi^{-1})(\mathrm{Nm}(j(x_i)) = (\chi_\nu\chi^{-1})(N_{F_\mathbb{A}^\times/\mathbb{Q}_\mathbb{A}^\times}(x_i)) = \eta(x_i)$. Thus

$$\mathrm{vol}(U_{F,c})^{-1}\varphi_{fin}(j(\alpha'))L_{\eta,j}(g') = (-2\alpha'\iota)^{-k} \sum_{i=1}^{h'} \varphi_{fin}(j(\alpha')^{x_i})P(g', j(\alpha')^{x_i}, \Gamma').$$

It is shown in Sec. 4.2 of [7] that $a_\xi(h) = C \cdot |\nu\xi|^{-1/2} S$ where $C = 6[U_0 : U_0(\chi)]^{-1} \prod_{q|N^+}(q+1)^{-1} \prod_{q|N^-}(q-1)^{-1}$ and

$$S = \sum_{\mathscr{C} \in R(\Gamma_\chi), Q(\mathscr{C}) = |\nu|\xi} \varphi_{fin}(x) P(g_\chi, x, \Gamma_\chi)$$

where x is any element of $\mathscr{C}$. For a fixed class $\mathscr{C}$, the choice of $x \in \mathscr{C}$ yields an embedding of $F = \mathbb{Q}(\sqrt{|\nu|\xi})$ in B sending $\sqrt{|\nu|\xi}$ to x. A different choice of x, say $x' \in \mathscr{C}$ corresponds to an embedding that is conjugate to the original one by an element of Γ_χ. Suppose $|\nu|\xi = d^2\xi'$ where ξ' is square-free, so that $F = \mathbb{Q}(\sqrt{\xi'})$ and that the embedding corresponding to x has conductor c. Then the conductor of the embedding corresponding to x' is also c, and hence we say that the conductor of $\mathscr{C}$ is c. Conversely, given any Γ_χ-conjugacy class of embeddings of conductor c, one has $\mathbb{O}' \cap F = \mathbb{Z} + \mathbb{Z}c\delta$ and $j(d\sqrt{\xi'})$ is an element of V, the class of which satisfies $Q(\mathscr{C}) = d^2\xi' = |\nu|\xi$.

For ξ' square-free, let $R(\Gamma', c, d, \xi')$ denote the set of $\mathscr{C} \in R(\Gamma')$ such that $Q(\mathscr{C}) = d^2\xi'$ and that have conductor c. There is a natural action of $B^\times_{\mathbb{A}_f}$ on $\bigcup_c R(\Gamma', c, d, \xi')$, that may be described as follows: for $g \in B^\times_{\mathbb{A}_f}$, write $g = \gamma \cdot u$ with $\gamma \in B^\times, u \in U'$ and define $\mathscr{C}^g = \gamma^{-1}\mathscr{C}\gamma$. This is easily seen to be independent of the choice of decomposition $g = \gamma \cdot u$. Indeed, if $g = \gamma \cdot u = \gamma_1 \cdot u_1$, then $\delta = \gamma_1^{-1}\gamma \in \Gamma'$, and $\delta^{-1}(\gamma_1^{-1}\alpha\gamma_1)\delta = \gamma^{-1}\alpha\gamma$. Clearly $\mathscr{C}^u = \mathscr{C}$ for any $u \in U'$.

Let $\mathbb{O}_{c,\xi'} = \mathbb{Z} + c\mathbb{Z}\delta \subset \mathbb{Q}(\sqrt{\xi'})$. The Picard group of $\mathbb{O}_{c,\xi'}$, $\mathrm{Pic}(\mathbb{O}_{c,\xi'})$ is identified naturally with $\hat{\mathbb{O}}^\times_{c,\xi'} \setminus F^\times_{\mathbb{A}_f}/F^\times$, hence acts (freely) on $R(\Gamma', c, d, \xi')$ via $\mathscr{C}^x := \mathscr{C}^{j(x)}$. Let $\tilde{R}(\Gamma', c, d, \xi')$ denote the set of equivalence classes for this action. Then

$$\begin{aligned} S &= \sum_c \sum_{\tilde{\mathscr{C}} \in \tilde{R}(\Gamma', c, d, \xi')} \sum_{\mathscr{C} \in \tilde{\mathscr{C}}} \varphi_{fin}(x) P(g_\chi, x, \Gamma_\chi) \\ &= (-2\alpha' i)^k \sum_c \mathrm{vol}(U_{F,c})^{-1} \sum_{\tilde{\mathscr{C}} \in \tilde{R}(\Gamma', c, d, \xi')} \varphi_{fin}(j_{\mathscr{C}}(\alpha')) L_{\eta, j_{\mathscr{C}}}(g_\chi) \end{aligned}$$

where $\alpha' = d\sqrt{\xi'}$ and $\mathscr{C}$ is any element in $\tilde{\mathscr{C}}$. Note that the value of the expression $\varphi_{fin}(j_{\mathscr{C}}(\alpha')) L_{\eta, j_{\mathscr{C}}}(g_\chi)$ is independent of the choice of $\mathscr{C} \in \tilde{\mathscr{C}}$.

Proposition 2.1. *If $L_{\eta,\mathscr{C}}(g_\chi) \neq 0$, the following conditions must be satisfied at all $q \mid N$:*
(a) *If $\chi_{0,q}(-1) = 1$, then $\left(\frac{\xi_0}{q}\right) \neq -w_q$.*
(b) *If $\chi_{0,q}(-1) = -1$, then $\left(\frac{\xi_0}{q}\right) = \chi_{0,q}(-1) w_q$.*

Proof. Fix $q \mid N$ and let ϵ_F denote the quadratic character corresponding to F. By the main result of [10], if $L_{\eta,\mathscr{C}}(g_\chi) \neq 0$, $\varepsilon_q(1/2, \pi_{f_\chi,F} \otimes \eta) = \alpha_q$, where $\pi_{f,F}$ denotes the base change of π_f to F and $\varepsilon_q(1/2, \cdot)$ denotes the epsilon factor at q. For simplicity, we omit the subscript q below. Define $\gamma_a = \pm 1$ by $\varepsilon(1/2, \pi_f \otimes \chi_a) = \gamma_a \chi_a(-1) \varepsilon(1/2, \pi_f)$ (where again, by χ_a

we mean $\chi_{a,q}$ etc.) Then

$$\begin{aligned}\varepsilon(1/2,\pi_{f_\chi,F}\otimes\eta) &= \epsilon_F(-1)\varepsilon(1/2,\pi_{f_\chi}\otimes \mathrm{Ind}_F^{\mathbb{Q}}\eta)\\ &= \chi_\nu\xi_0(-1)\varepsilon(1/2,\pi_{f_\chi}\otimes(\chi_\nu\chi^{-1}))\varepsilon(1/2,\pi_{f_\chi}\otimes(\chi_\nu\chi^{-1}\epsilon_F))\\ &= \chi_\nu\xi_0(-1)\varepsilon(1/2,\pi_f\otimes\chi_\nu)\varepsilon(1/2,\pi_f\otimes\chi_{\xi_0})\\ &= \chi_\nu\xi_0(-1)\cdot\gamma_\nu\chi_\nu(-1)\varepsilon(1/2,\pi_f)\cdot\chi_{\xi_0}(-1)\gamma_{\xi_0}\varepsilon(1/2,\pi_f)\\ &= \gamma_\nu\gamma_{\xi_0}.\end{aligned}$$

We now consider four cases (and use (1.1) as well as Lemma 3.1 of [7]):

(i) If $q\nmid\nu$, $q\nmid\xi$, $\gamma_\nu=\chi_\nu(q)$, $\gamma_{\xi_0}=\chi_{\xi_0}(q)$, $\chi_\nu(q)=\alpha\chi_0(-1)w$, thus $\alpha=\alpha\chi_0(-1)$ $w\left(\frac{\xi_0}{q}\right)$, whence $\left(\frac{\xi_0}{q}\right)=\chi_0(-1)w$.

(ii) If $q\nmid\nu$, $q\mid\xi$, $\gamma_\nu=\chi_\nu(q)$, $\gamma_{\xi_0}=w$, $\chi_\nu(q)=\alpha\chi_0(-1)w$, whence $\chi_0(-1)=1$ and $\left(\frac{\xi_0}{q}\right)=0$.

(iii) If $q\mid\nu$, $q\nmid\xi$, $\gamma_\nu=w$, $\gamma_{\xi_0}=\chi_{\xi_0}(q)$, $\chi_0(-1)=\alpha$, whence $\left(\frac{\xi_0}{q}\right)=\chi_0(-1)w$.

(iv) If $q\mid\nu$, $q\mid\xi$, $\gamma_\nu=w$, $\gamma_{\xi_0}=w$, $\chi_0(-1)=\alpha$. Hence $\chi_0(-1)=1$ and $\left(\frac{\xi_0}{q}\right)=0$.

The proposition readily follows from (i)–(iv). □

Proposition 2.2. *If $S\neq 0$, then the following conditions must be satisfied at all $q\mid N$:*
(a) *For all $q\mid N, q\nmid N'$, $\left(\frac{\xi_0}{q}\right)\neq -w_q$.*
(b) *For all $q\mid N'$, $\left(\frac{\xi_0}{q}\right)=\chi_{0,q}(-1)w_q$.*
(c) $\xi_0\equiv 0,1 \mod 4$.

Proof. Part (c) follows immediately from the definition of the Schwartz function φ_2. By the previous proposition, to prove parts (a) and (b), it suffices to show that $S=0$ if there exists $q\mid N'$ such that $q\mid\xi$ and $\chi_{0,q}(-1)=1$. In this case $q\nmid\nu$ by (iv) of the previous proposition and the assumption (following (1.1) in the introduction) that $\chi_{0,q}$ is unramified if $q\mid N^+, q\mid\nu$.

If further $q\mid N^+$, we are in case (c2) of the definition of φ. Let $\mathcal{O}_1$ be the Eichler order of level q, $\begin{pmatrix}\mathbb{Z}_q & \frac{1}{q}\mathbb{Z}_q\\ q^2\mathbb{Z}_q & \mathbb{Z}_q\end{pmatrix}$. Then S is a linear combination of terms of the form

$$(2.4)\qquad S'=\sum_{u\in\mathcal{O}_1^\times/\mathcal{O}'^\times_q}\varphi_{fin}(j_{\mathscr{C}^u}(\alpha'))L_{\eta,j_{\mathscr{C}^u}}(g_\chi).$$

It is thus enough to show that the expression (2.4) equals 0. Note that the character $\tilde{\omega}_\chi$ can be extended to $\mathcal{O}_1^\times$ by defining $\tilde{\omega}_\chi\begin{pmatrix}a&b\\c&d\end{pmatrix}=\chi_q^2(d)$. Thus χ' extends too to the group $\Gamma_1=(\prod_{l\neq q}U_{0,l}(\chi)\times\mathcal{O}_1^\times)\cap B^\times$. Now a set of representatives for $\mathcal{O}_1^\times/\mathcal{O}'^\times_q$ is $\left\{\begin{pmatrix}1&j/q\\0&1\end{pmatrix}\right\}$, $j=0,1,\dots,q-1$. For $u'\in\mathcal{O}_1^\times/\mathcal{O}'^\times_q$, write $u'=\gamma_u\cdot u$ where $\gamma_u\in B^\times$ and $u\in U_0(\chi)$, so that $\gamma_u \in \Gamma_1$. For any $x \in \mathscr{C}$, $\varphi_{fin}(\gamma_u^{-1} x \gamma_u) = \varphi_{fin}(uu'^{-1}xu'u^{-1}) = \tilde{\omega}_\chi(u)\varphi_{fin}(u'^{-1}xu')$. Now set $x = j_{\mathscr{C}}(\alpha')$. If $\varphi(x) \neq 0$, we must have

$$x_q = \begin{pmatrix} b & -a \\ c & -b \end{pmatrix}$$

with $v_q(a) = -1$, $v_q(b) \geq 0$ and $v_q(c) \geq 2$. However, $q \mid \xi$, hence $v_q(b) \geq 1$ as well. Let $a = a'/q$. Since

$$\begin{pmatrix} 1 & \frac{j}{q} \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} b & -a \\ c & -b \end{pmatrix} \begin{pmatrix} 1 & \frac{j}{q} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} b - j\frac{c}{q} & \frac{-cj^2/q + 2bj - a'}{q} \\ c & j\frac{c}{q} - b \end{pmatrix}$$

we have $\varphi_{fin}(u'^{-1}xu') = \varphi_{fin}(x)$ for $u' = \left(\begin{smallmatrix} 1 & j/q \\ 0 & 1 \end{smallmatrix}\right)$. Thus $\varphi_{fin}(u'^{-1}xu') = \varphi_{fin}(x)\tilde{\omega}_\chi(u')^{-1}$ for all $u' \in \mathcal{O}_1^\times$ and

$$\begin{aligned} S' &= \sum_{u' \in \mathcal{O}_1^\times / \mathcal{O}'^\times_q} \varphi_{fin}(\gamma_u^{-1} x \gamma_u) L_{\eta, \gamma_u^{-1} j_{\mathscr{C}} \gamma_u}(g_\chi) \\ &= \varphi_{fin}(x) \sum_{u' \in \mathcal{O}_1^\times / \mathcal{O}'^\times_q} \tilde{\omega}_\chi(u')^{-1} \tilde{\omega}_\chi(u) L_{\eta, \gamma_u^{-1} j_{\mathscr{C}} \gamma_u}(g_\chi) \\ &= \varphi_{fin}(x) L_{\eta, j_{\mathscr{C}}}(F) \end{aligned}$$

where $F(\cdot) = \sum_{u' \in \mathcal{O}_1^\times / \mathcal{O}'^\times_q} g_\chi(\cdot u')\tilde{\omega}_\chi^{-1}(u')$. Since g_χ is a newform, and $F(\cdot u) = \tilde{\omega}_\chi(u)F(\cdot)$ for all $u \in \mathcal{O}_1^\times$, we must have that $F = 0$, whence $S' = 0$ and $S = 0$ as required.

If $q \mid N^-$, let $j(\alpha') = a + bu \in \mathcal{O}' \otimes \mathbb{Z}_q$. Then $N(a) - qN(b) = \mathrm{Nm}(a + bu) = -\nu\xi \Rightarrow v_q(N(a) - qN(b)) = 1 \Rightarrow v_q(a) \neq 0 \Rightarrow \varphi_q(j(\alpha')) = 0$, hence $S = 0$ in this case as well. □

It follows from the proposition above that h is a scalar multiple of the form $h_{f,\chi}$ defined in [7] Prop. 2.3. We shall show below that it is in fact a nonzero scalar multiple of $h_{f,\chi}$ by computing the absolute value of the ξth Fourier coefficient for ξ_0 a fundamental discriminant. We assume henceforth that ξ_0 is a fundamental quadratic discriminant.

Let $Emb_q(\Gamma', c, \xi')$ denote the set of optimal embeddings of $\mathcal{O}_{c,\xi',q}$ into $\mathcal{O}'_q$ modulo the conjugation action of $\mathcal{O}'^\times_q$. Also let E_q be the subset of $\bigcup_c Emb_q(\Gamma', c, \xi')$ consisting of classes $[j]$ such that $\varphi_q(j(d\sqrt{\xi'})) \neq 0$ and set $e_q = |E_q|$. Let us recall the following result of Hijikata, which we quote verbatim but specialized to our situation. In the statement below, $\mathfrak{r} = \mathbb{Z}_q =$ the maximal order in $\mathbb{Q}_q$ and $\pi =$ a uniformizer in $\mathfrak{r}$.

Theorem 2.3 ([4] Thm. 2.3). *Suppose B_q is split at q and $\mathcal{O}'_q = \left(\begin{smallmatrix} \mathfrak{r} & \mathfrak{r} \\ q^v\mathfrak{r} & \mathfrak{r} \end{smallmatrix}\right)$. Let F_q be a quadratic extension of $\mathbb{Q}_q$, either split or not, g an integral element of F_q with $F_q = \mathbb{Q}_q + \mathbb{Q}_q g$ and set $\Lambda = \mathfrak{r} + \mathfrak{r}g$. Let $f(X) = X^2 - sX + n \in \mathfrak{r}[X]$ be the minimal polynomial of g over $\mathbb{Q}_q$. Put $\tilde{R} = \{\kappa \in \mathfrak{r} \mid f(\kappa) \equiv 0 \mod q^v\}$. Let R be a complete set of representatives for $\tilde{R}$ modulo q^v and put $R' = \{\kappa \in R \mid f(\kappa) \equiv 0 \mod q^{v+1}\}$. Define $\phi_x : F_q \to B_q$ by $\phi_x(g) = \left(\begin{smallmatrix} x & 1 \\ -f(x) & s-x \end{smallmatrix}\right)$ for $x \in R$ and $\phi'_x : F_q \to B_q$ by $\phi'_x(g) = \left(\begin{smallmatrix} s-x & -q^{-v}f(x) \\ q^v & x \end{smallmatrix}\right)$ for $x \in R'$.*

If $s^2 - 4n$ is a unit of $\mathfrak{r}$ *or* $\upsilon = 0$ *(resp. not a unit of* $\mathfrak{r}$ *and* $\upsilon > 0$*), the set* $\{\phi_x \mid x \in R\}$ *(resp.* $\{\phi_x \mid x \in R\} \cup \{\phi'_x | x \in R'\}$*) is a complete set of representatives for the set of optimal embeddings of* Λ *in* $\mathbb{O}'_q$ *modulo the conjugation action of* $\mathbb{O}'^{\times}_q$*.*

We now use the theorem stated above (with $\Lambda = \mathbb{O}_{c,\xi',q}$) to study the sets E_q in each of the cases that occurs in the definition of φ.

Case (a): $\upsilon = 0, (d) = 1$. We must have $(c) = 1$. Setting $g = \sqrt{\xi'}$, we may take $E = \{j\}$, where $j(g) = \begin{pmatrix} 0 & 1 \\ \xi' & 0 \end{pmatrix}$ and $\varphi_q(j) = 1$.

Case (b): $\upsilon = 0$. If $q \nmid \xi$, $(d) = 1$ and $(c) = 1$. Setting $g = \sqrt{\xi'}$, we may take $E = \{j\}$, where $j(g) = \begin{pmatrix} 0 & 1 \\ \xi' & 0 \end{pmatrix}$ and $\varphi_q(j) = 1$. If $q \mid \xi$, $(d) = (q)$ and necessarily $(c) = (q)$. Setting $g = q\sqrt{\xi'}$, we may take $E = \{j\}$, where $j(g) = \begin{pmatrix} 0 & 1 \\ q^2\xi' & 0 \end{pmatrix}$ satisfies $\varphi_q(j) = 1$.

Case (c1): $\upsilon = 1$, $(d) = 1$. Necessarily, $(c) = 1$. Take $g = \sqrt{\xi'}$ so that $s = 0, n = -\xi'$. $\tilde{R} = \{\kappa \in \mathfrak{r}, \kappa^2 - \xi' \equiv 0 \mod q\}$. If $q \mid \xi', R = \{0\}, R' = \emptyset, j_0(g) = \begin{pmatrix} 0 & 1 \\ \xi' & 0 \end{pmatrix}, \varphi_q(j_0(g)) = 1$. If $q \nmid \xi$, $R = \emptyset$ if F_q is inert, while if F_q is split, $R = \{x_1, x_2\}, \overline{x}_1, \overline{x}_2$ being the two distinct roots of ξ' mod $q, j_{x_i}(g) = \begin{pmatrix} x_i & 1 \\ \xi' - x_i^2 & -x_i \end{pmatrix}$, and $\varphi_q(j_{x_i}(g)) = 1$.

Case (c2): $\upsilon = 2$, $(d) = 1$. Necessarily, $(c) = q$. Let $g = q\sqrt{\xi'}$ so that $s = 0, n = -q^2\xi'$. $\tilde{R} = \{\kappa \in \mathfrak{r}, \kappa^2 - q^2\xi' \equiv 0 \mod q^2\}$, $R = q\mathbb{Z}_q/q^2\mathbb{Z}_q$. For $x = qi \in R$, $j_x(g) = \begin{pmatrix} x & 1 \\ q^2\xi' - x^2 & -x \end{pmatrix}$.

We only need to consider the case $q \nmid \xi'$. If further F_q is split, $\varphi_q(j_x(\sqrt{\xi'})) = 0$ unless $x \in \{qx_1, qx_2\}$ in which case $\varphi_q(j_{x_i}(\sqrt{\xi'})) = 1$. Also, $\varphi_q(j(\sqrt{\xi'})) = 0$ for all $j \in R'$. If on the other hand, F_q is inert, $\varphi_q(j_x(\sqrt{\xi'})) = 0$, and $R' = \emptyset$.

Case (c3): $\upsilon = 1$. If $q \mid \xi$, $(d) = q$, and necessarily $(c) = (q)$. Take $g = q\sqrt{\xi'}$, so that $s = 0, n = -q^2\xi'$. $\tilde{R} = \{\kappa \in \mathfrak{r}, \kappa^2 - q^2\xi' \equiv 0 \mod q\}$. $R = \{0\}, j_0(g) = \begin{pmatrix} 0 & 1 \\ q^2\xi' & 0 \end{pmatrix}$ and $\varphi_q(j_0(q\sqrt{\xi'})) = 1$. $R' = \{0\}, j'_0(g) = \begin{pmatrix} 0 & q\xi' \\ q & 0 \end{pmatrix}, \varphi_q(j'_0(q\sqrt{\xi'})) = 0$.

If $q \nmid \xi$, $(d) = 1$ and $(c) = 1$. Take $g = \sqrt{\xi'}$, so that $s = 0$ and $n = -\xi'$. $\tilde{R} = \{\kappa \in \mathfrak{r}, \kappa^2 - \xi' \equiv 0 \mod q\}$. $R = \{0\}, j_0(g) = \begin{pmatrix} 0 & 1 \\ \xi' & 0 \end{pmatrix}$ and $\varphi_q(j_0(\sqrt{\xi'})) = 1$. $R' = \emptyset$.

Case (d1): Suppose first that $\chi_{0,q}$ is unramified. We must have $(c) = 1$. Clearly E_q is nonempty if and only if F_q is not split. If F_q is ramified, E_q consists of a single element j with $\varphi_q(j(\sqrt{\xi'}) = 1$. If F_q is inert, E_q consists of two elements j_1, j_2 with $\varphi_q(j_i(\sqrt{\xi'})) = 1$.

Next suppose $\chi_{0,q}$ is ramified. Again $(c) = 1$. Now it is clear that E is nonempty exactly when F_q is inert at q; in this case, E_q consists of two elements j_1, j_2 with $\varphi_q(j_i(\sqrt{\xi'}))$ being complex numbers of absolute value 1.

Case (d2): Again $d = 1$ and necessarily $(c) = 1$; E_q is nonempty exactly when $q \nmid \xi$ and $(\xi_0, q) = -w_q$. In that case, F_q is ramified and E_q consists of a single element j with $\varphi_q(j)$ a complex number of absolute value 1.

Case (e): $q = 2$, $v = 0$ and $R = \{0\}$. If $\xi_0 \equiv 1 \mod 4$, then $\xi' \equiv 1 \mod 4$ as well and $(d) = (1)$. Pick $g = c\frac{1+\sqrt{\xi'}}{2}$, so that $s = c$ and $n = c^2(1-\xi')/4$. Then $j_0(\sqrt{\xi'})) = \begin{pmatrix} -1 & 2c^{-1} \\ c\frac{\xi'-1}{2} & 1 \end{pmatrix}$. Thus $\varphi_2(j_0(d\sqrt{\xi'})) \neq 0 \iff (c) = (1)$ and in that case it equals 1. If $\xi_0 \equiv 4$ (resp.) $0 \mod 8$, then $\xi' \equiv 3$ (resp.) $2 \mod 8$, $(d) = (2)$ and we may pick $g = c\sqrt{\xi'}$. Then $j_0(d\sqrt{\xi'}) = \begin{pmatrix} 0 & c^{-1}d \\ cd\xi' & 0 \end{pmatrix}$. Again, $\varphi_2(j_0(d\sqrt{\xi'})) \neq 0 \iff (c) = (1)$ and is equal to 1 in this case.

Note that the only primes q for which E_q has cardinality greater than 1 are primes of the form (c1) or (c2) with F_q split or those of the form (d1) with F_q inert; in each of these cases $|E_q| = 2$. The following lemma shows that it is sufficient to consider any fixed embedding j such that $j_q \in E_q$.

Lemma 2.4. *Let $j = j_{\mathscr{C}}$ and $j' = j_{\mathscr{C}'}$ be two embeddings satisfying $j_q, j'_q \in E_q$ for all q. Then j and j' have the same conductor and*

$$\varphi_{fin}(j_{\mathscr{C}}(\alpha'))L_{\eta,j_{\mathscr{C}}}(g_\chi) = \varphi_{fin}(j_{\mathscr{C}'}(\alpha'))L_{\eta,j_{\mathscr{C}'}}(g_\chi).$$

Proof. It is enough to consider the case where $[j_q] \neq [j'_q]$ for exactly one q. Without loss we may also assume that the LHS in the relation above is not zero. As noted above, q must be a prime of the form (c1) or (c2) with F_q split or of the form (d1) with F_q inert. Let us pick an element $g_q \in B_q^\times$ such that $j'_q = g_q^{-1} \cdot j_q \cdot g_q$ and denote by δ the element of $B_{\mathbb{A}_f}^\times$ whose component at q is g_q and all whose other components are 1. Then $[j'] = [j]^\delta$. Suppose $\delta = \gamma \cdot u$ with $\gamma \in B^\times$ and $u \in U'$, so that without loss we may assume $j' = \gamma^{-1} j \gamma$. Now

$$\begin{aligned}
\varphi_{fin}(j'(\alpha')) &= \varphi_{fin}(\gamma^{-1} j(\alpha')\gamma) = \varphi_{fin}(u\delta^{-1} j(\alpha)\delta u^{-1}) \\
&= (\tilde{\omega}_\chi \cdot ((\chi_\nu\chi) \circ \mathrm{Nm})^{-1})(u) \cdot \varphi_{fin}(\delta^{-1} j(\alpha)\delta) \\
&= \tilde{\omega}_\chi(u) \cdot (\chi_\nu\chi)^{-1}(\mathrm{Nm}\,\delta) \cdot \frac{\varphi_q(g_q^{-1} j_q(\alpha) g_q)}{\varphi_q(j_q(\alpha))} \cdot \varphi_{fin}(j(\alpha)).
\end{aligned}$$

Also letting $\gamma'_\infty = \gamma^{-1}\gamma_\infty$ and considering $F^\times$ as a subgroup of $B^\times$ via j,

$$\begin{aligned}
L_{\eta,j'}(g_\chi) &= \int_{F^\times \mathbb{Q}_\mathbb{A}^\times \backslash F_\mathbb{A}^\times} s_{g_\chi}(j'(x)\gamma'_\infty)\eta(x)d^\times x \\
&= \int_{F^\times \mathbb{Q}_\mathbb{A}^\times \backslash F_\mathbb{A}^\times} s_{g_\chi}(\gamma^{-1} x \gamma \gamma'_\infty)\eta(x)d^\times x \\
&= \int_{F^\times \mathbb{Q}_\mathbb{A}^\times \backslash F_\mathbb{A}^\times} s_{g_\chi}(x\delta u^{-1}\gamma_\infty)\eta(x)d^\times x \\
&= \tilde{\omega}_\chi(u)^{-1} \int_{F^\times \mathbb{Q}_\mathbb{A}^\times \backslash F_\mathbb{A}^\times} s_{g_\chi}(x g_q \gamma_\infty)\eta(x)d^\times x.
\end{aligned}$$

Thus

$$\frac{\varphi_{fin}(j'(\alpha'))L_{\eta,j'}(g_\chi)}{\varphi_{fin}(j(\alpha'))L_{\eta,j}(g_\chi)} = (\chi_\nu\chi)^{-1}(\mathrm{Nm}\,\delta) \cdot \frac{\varphi_q(g_q^{-1} j_q(\alpha) g_q)}{\varphi_q(j_q(\alpha))} \cdot \frac{\int_{F^\times \mathbb{Q}_\mathbb{A}^\times \backslash F_\mathbb{A}^\times} s_{g_\chi}(x g_q \gamma_\infty)\eta(x)d^\times x}{\int_{F^\times \mathbb{Q}_\mathbb{A}^\times \backslash F_\mathbb{A}^\times} s_{g_\chi}(x\gamma_\infty)\eta(x)d^\times x}.$$

Let λ be such that

$$\lambda\varphi_{fin}(j_{\mathscr{C}}(\alpha'))L_{\eta,j_{\mathscr{C}}}(g_\chi) = \varphi_{fin}(j_{\mathscr{C}'}(\alpha'))L_{\eta,j_{\mathscr{C}'}}(g_\chi).$$

We now consider separately the following cases:

Case (c1): We can take $j_q(\alpha') = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, j'_q(\alpha') = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$ where $i^2 = \xi'$. Then j_q and j'_q are conjugate by $g_q = \begin{pmatrix} 0 & -1 \\ q & 0 \end{pmatrix}$. Also $s_{g_\chi}(xg_q) = w_q\chi_q(q)s_{g_\chi}(x)$, $\varphi_q(g_q^{-1}j_q(\alpha)g_q) = \varphi_q(j_q(\alpha))$ and $\chi_\nu\chi(\mathrm{Nm}\,\delta) = \chi_{\nu,q}(q)\chi_q(q)$. Thus $\lambda = w_q\chi_{\nu,q}(q) = 1$ by (1.1).

Case (d1): We can take $g_q = u$ where u is as in [7] Sec. 3.2. Then $\varphi_q(g_q^{-1}j_q(\alpha)g_q) = \chi_q(-1)\varphi_q(j_q(\alpha))$ and $s_{g_\chi}(xg_q) = w'_q\chi_q(\mathrm{Nm}\,u_0)s_{g_\chi}(x)$, so that $\lambda = \chi_{0,q}(-1)\chi_{\nu,q}(q)w'_q = 1$, again by (1.1).

Case (c2): We may assume that $j_q(\alpha') = \begin{pmatrix} i & 1/q \\ 0 & -i \end{pmatrix}$ and $j'_q(\alpha') = \begin{pmatrix} -i & 1/q \\ 0 & i \end{pmatrix}$ where $i^2 = \xi'$. Let $x = j_{\mathscr{C}}(\alpha')$. Now we see from (2.5) that for $u' = u'_k := \begin{pmatrix} 1 & k/q \\ 0 & 1 \end{pmatrix} \in \mathcal{O}_1^\times/\mathcal{O}'^\times_q$, $j_{\mathscr{C}u'}$ is conjugate to $j_{\mathscr{C}}$ by U' exactly when $2ik+1 \not\equiv 0 \mod q$. Let β_r be the idele whose q component is $\begin{pmatrix} 1 & r/q \\ 0 & 1 \end{pmatrix}$ and all whose other components are 1. Note that $\beta_{1/2i}^{-1}\begin{pmatrix} i & 1/q \\ 0 & -i \end{pmatrix}\beta_{1/2i} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$. Let $j_0 = \gamma^{-1}j_{\mathscr{C}}\gamma$ where $\beta_{1/2i} = \gamma u$ for $\gamma \in B^\times$ and some u in a deep enough subgroup of U'. Then

$$\begin{aligned}
(q-1)\varphi_{fin}(j(\alpha'))L_{\eta,j}(g_\chi) &= \sum_{\substack{u'=\begin{pmatrix} 1 & k/q \\ 0 & 1 \end{pmatrix}\in\mathcal{O}_1^\times/\mathcal{O}'^\times_q \\ 2ik+1\not\equiv 0 \mod q}} \varphi_{fin}(j_{\mathscr{C}u'}(\alpha'))L_{\eta,j_{\mathscr{C}u'}}(g_\chi) \\
&= \sum_{\substack{k=0 \\ 2ik+1\not\equiv 0 \mod q}}^{q-1} \chi_q^{-1}(2ik+1)\int_{F^\times\mathbb{Q}_{\mathbb{A}}^\times\backslash F_{\mathbb{A}}^\times} s_{g_\chi}(j_{\mathscr{C}}(x)\beta_k\gamma_\infty)\eta(x)d^\times x \\
&= \int_{F^\times\mathbb{Q}_{\mathbb{A}}^\times\backslash F_{\mathbb{A}}^\times} \sum_{\substack{k=0 \\ 2ik+1\not\equiv 0 \mod q}}^{q-1} \chi_q^{-1}(2ik+1)s_{g_\chi}(j_0(x)\beta_{1/2i}\beta_k\gamma'_\infty)\eta(x)d^\times x \\
&= \chi_q^{-1}(2iq)\int_{F^\times\mathbb{Q}_{\mathbb{A}}^\times\backslash F_{\mathbb{A}}^\times} \sum_{k'=1}^{q-1} \chi_q^{-1}(k')s_{g_\chi}(j_0(x)\beta_{k'}\gamma'_\infty)\eta(x)d^\times x
\end{aligned}$$

where γ'_∞ satisfies ${\gamma'_\infty}^{-1}j_0(\alpha')\gamma'_\infty = \begin{pmatrix} \alpha & 0 \\ 0 & -\alpha \end{pmatrix}$. Let δ_q (resp. δ'_q) $\in B_{\mathbb{A}_f}^\times$ be the element whose component at q is $\begin{pmatrix} 1/q & 0 \\ 0 & 1 \end{pmatrix}$ (resp. $\begin{pmatrix} 0 & -1 \\ q & 0 \end{pmatrix}$) and all whose other components are 1. Since $s_{g_\chi}(\cdot) = G(\chi,\psi)^{-1}\sum_{l=1}^{q-1}\chi_q(l)s_{g'}(\cdot\beta_l)$, where $s_{g'}$ is a new form in $s_{g_\chi}\otimes\chi_q^{-1}$, we have

$$G(\chi,\psi)\sum_{k'=1}^{q-1}\chi_q^{-1}(k')s_{g_\chi}(\cdot\beta_{k'}) = \sum_{k'=1}^{q-1}\sum_{l=1}^{q-1}\chi_q^{-1}(k')\chi_q(l)s_{g'}(\cdot\beta_{k'+l})$$
$$= \chi_q(-1)\chi_q(\mathrm{Nm}(\cdot))\Big\{(qs_{g'}(\cdot)) - \sum_{j=0}^{q-1} s_{g'}(\cdot\beta_j)\Big\}$$
$$= \chi_q(-1)\chi_q(\mathrm{Nm}(\cdot))\left\{(qs_{g'}(\cdot)) + w_q s_{g'}(\cdot\delta_q)\right\}.$$

Thus, $G(\chi,\psi)(q-1)\varphi_{fin}(j(\alpha'))L_{\eta,j}(g_\chi)$

$$= \chi_q^{-1}(-2iq)\int_{F^\times\mathbb{Q}_{\mathbb{A}}^\times\backslash F_{\mathbb{A}}^\times}\chi_\nu(\mathrm{Nm}(x))\left\{qs_{g'}(j_0(x)\gamma'_\infty) + w_q s_{g'}(j_0(x)\delta_q\gamma'_\infty)\right\}d^\times x$$
$$= \chi_q^{-1}(-2iq)(q + w_q\chi_{\nu,q}(q))\int_{F^\times\mathbb{Q}_{\mathbb{A}}^\times\backslash F_{\mathbb{A}}^\times}\chi_\nu(\mathrm{Nm}(x))s_{g'}(j_0(x)\gamma'_\infty)d^\times x.$$

Since $\begin{pmatrix}0&-1\\q&0\end{pmatrix}^{-1}\begin{pmatrix}i&0\\0&-i\end{pmatrix}\begin{pmatrix}0&-1\\q&0\end{pmatrix} = \begin{pmatrix}-i&0\\0&i\end{pmatrix}$, $s_{g'}(\cdot\delta'_q) = w_q s_{g'}(\cdot)$ and $\chi_q(-1)w_q\chi_{\nu,q}(q) = 1$, we see that

$$\varphi_{fin}(j'(\alpha'))L_{\eta,j'}(g_\chi) = \chi_q(-1)w_q\chi_{\nu,q}(q)\varphi_{fin}(j(\alpha'))L_{\eta,j}(g_\chi) = \varphi_{fin}(j(\alpha'))L_{\eta,j}(g_\chi).$$

Thus, in any case, $\lambda = 1$ as required. □

The lemma shows that $|S| = |2\alpha'|^k\mathrm{vol}(U_{F,c}^{-1})\cdot 2^t|L_{\eta,j}(g_\chi)|$ for any choice of j such that $j_q \in E_q$ for all q, where t is the number of primes for which $|E_q| = 2$. This yields the following preliminary proposition.

Proposition 2.5. *Suppose ξ_0 is a fundamental quadratic discriminant satisfying conditions* (a), (b), (c) *of Prop. 2.2. Then*

$$|a_\xi(h)|^2 = 4^{k+t}C^2|\nu\xi|^{k-1}\mathrm{vol}\,(U_{F,c})^{-2}|L_{\eta,j}(g_\chi)|^2$$

for any choice of embedding j such that $j_q \in E_q$ for all q and for c equal to the conductor of j. Here $C = 6[U_0 : U_0(\chi)]^{-1}\prod_{q|N^+}(q+1)^{-1}\prod_{q|N^-}(q-1)^{-1}$.

The problem is then reduced to computing the quantity $|L_{\eta,j}(g_\chi)|^2$ to which we apply the method of Waldspurger as outlined in the next section.

3 The method of Waldspurger

3.1 Quaternary forms: the dual pair $(\mathrm{GL}_2, \mathrm{GO}(B))$

Let π be the automorphic representation of $\mathrm{GL}_2(\mathbb{A})$ associated to f, π_χ the representation associated to f_χ and $\pi' = JL(\pi)$, $\pi'_\chi = JL(\pi_\chi)$ the corresponding representations of $B^\times$. Let ψ be the usual additive character on $\mathbb{Q}\setminus\mathbb{A}_\mathbb{Q}$ and $\Theta^t(\pi_\chi) := \Theta^t(\pi_\chi,\psi)$ the theta lift of π_χ to $\mathrm{GO}(B)$. Denote by $\tilde{\Theta}(\pi_\chi)$ the pull back of $\Theta^t(\pi_\chi)$ via the natural map $B^\times(\mathbb{A})\times B^\times(\mathbb{A})\to \mathrm{GO}(B), (a,b)\mapsto (x \rightsquigarrow axb^{-1})$. It is a theorem of Shimizu that $\tilde{\Theta}(\pi_\chi)\simeq \pi'_\chi\otimes\pi'^{\vee}_\chi$. One may also pick an explicit Schwartz function $\varsigma\in\mathcal{S}(B(\mathbb{A}))$ such that $\tilde{\theta}^t(\varsigma,f_\chi)$ is a scalar multiple of $g_\chi \times \overline{g_\chi}$. Indeed such a function has been described in work of Watson [15] at all places where χ is not ramified. At finite places q with χ_q unramified, we take $\varsigma_q = \mathbb{I}_{\mathbb{O}_N \otimes \mathbb{Z}_q}$ (as opposed to Watson's choice of $\frac{1}{\mathrm{vol}(U_{0,q})}\mathbb{I}_{\mathbb{O}_N \otimes \mathbb{Z}_q}$). At ∞, we make the same choice of Schwartz function as in [15]. Finally, if χ_q is ramified, we make the following choice for ς_q.

Definition 3.1. If B is split at q, and χ^2 is ramified at q,

$$\varsigma_q \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \chi_q^{-2}(d)\mathbb{I}_{\mathbb{Z}_q}(a)\mathbb{I}_{\mathbb{Z}_q}(b)\mathbb{I}_{q^2\mathbb{Z}_q}(c)\mathbb{I}_{\mathbb{Z}_q^\times}(d).$$

If B is split at q, and χ^2 is unramified at q,

$$\varsigma_q \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \mathbb{I}_{\mathbb{Z}_q}(a)\mathbb{I}_{\mathbb{Z}_q}(b)\mathbb{I}_{q^2\mathbb{Z}_q}(c)\mathbb{I}_{\mathbb{Z}_q}(d).$$

If B is ramified at q,

$$\varsigma_q(u) = \chi_q^{-1}(\mathrm{Nm}(u))I_{\mathbb{O}_q^\times}(u).$$

By a similar argument as in [15], one sees that with this choice of ς (with $\langle,\rangle$ being the usual Petersson inner product).

Proposition 3.2.

$$\tilde{\theta}^t(\varsigma, f_\chi) = C' \cdot \frac{\langle f_\chi, f_\chi \rangle}{\langle g_\chi, g_\chi \rangle} \cdot (g_\chi \times \overline{g_\chi})$$

for $C' = \prod_{q<\infty} \mathrm{vol}(U_{0,q}(\chi))$.

3.2 A formula for the torus integral as a product of local zeta integrals

Let us choose τ such that $B = F + F\tau$, $\tau \in N_{B^\times}(F^\times) \setminus F^\times$, $\tau x = x^i \tau$ for $x \in j(F)$. Set $V_1 = F, V_2 = V_1^\perp = F\tau$ and suppose that $\bar{\varsigma}_q = \sum_{i \in I_q} \varsigma_{1,i} \otimes \varsigma_{2,i} \in \mathcal{S}(V_{1,q}) \otimes \mathcal{S}(V_{2,q})$. By an application of see-saw duality (and accounting for the measure normalizations),

$$\begin{aligned}
&\frac{\langle f_\chi, f_\chi \rangle}{\langle g_\chi, g_\chi \rangle} L_\eta(g_\chi) L_{\bar{\eta}}(\overline{g_\chi}) \\
&= \frac{\langle f_\chi, f_\chi \rangle}{\langle g_\chi, g_\chi \rangle} \langle g_\chi \times \overline{g_\chi}|_{F^\times(\mathbb{A}) \times F^\times(\mathbb{A})}, \bar{\eta} \times \eta \rangle \\
&= \frac{1}{C'} \langle \tilde{\theta}^t(\varsigma, f_\chi)|_{F^\times(\mathbb{A}) \times F^\times(\mathbb{A})}, \bar{\eta} \times \eta \rangle \\
&= \frac{1}{C'} \langle \theta^t(\varsigma, f_\chi)|_{G(O(V_1) \times O(V_2))^0(\mathbb{A})}, 1 \times \bar{\eta} \rangle \\
&= \frac{\sqrt{d_F} L(1, \epsilon_F)}{2C'} \cdot \sum_{(i_q) \in \prod_q I_q} \int_{GL_2(\mathbb{Q})Z(\mathbb{A}) \backslash GL_2(\mathbb{A})} f_\chi(\sigma) \theta_{\varsigma_{2,i_q}}(\eta)(\sigma) \theta_{\varsigma_{1,i_q}}(1)(\sigma) d^\times \sigma
\end{aligned}$$

where the measure in the last expression is the Tamagawa measure. Note that we have used above that η factors through the norm. Now using the fact that the theta lift of the trivial character is an Eisenstein series by the Siegel-Weil formula, and unfolding this Eisenstein series, one sees that the sum in the last expression above equals the value at $s = 1/2$ of the analytic continuation of

$$\zeta(2)^{-1}\prod_q \sum_{i\in I_q} \int_{\mathbb{Q}_q^\times}\int_{K_0} W_{f_\chi,q}(\mathbf{d}(a)k)W_{\eta,q}(\varsigma_{2,i},\mathbf{d}(-a)k)\Phi_q^s(\varsigma_{1,i},\mathbf{d}(-a)k)|a|^{-1}d^\times a dk$$

where $\Phi_q^s(\varsigma_1,\sigma) = |a(\sigma)|^{s-1/2}r_\psi(\sigma,h_0)\varsigma_1(0)$ and

$$W_{\eta,q}(\varsigma_2,\sigma) = |\operatorname{Nm}(\tau)|^{1/2}\int_{F_q^{(1)}} r_\psi\left(\begin{pmatrix}\operatorname{Nm}(\tau)^{-1} & 0\\ 0 & 1\end{pmatrix}\sigma, hh_0\right)\varsigma_2(\tau)\eta(hh_0)d^\times h$$

for any h_0 with $\operatorname{Nm}(h_0) = \operatorname{Nm}(\tau)^{-1}\det\sigma$ (and is 0 if $\operatorname{Nm}(\tau)^{-1}\det\sigma \notin \operatorname{Nm}(F_{\mathbb{A}}^\times)$). Here the measure $d^\times a$ is chosen such that $\operatorname{vol}(\mathbb{Z}_q^\times) = 1$ and dk such that $\operatorname{vol}(\mathrm{GL}_2(\mathbb{Z}_q)) = 1$ and $\operatorname{vol}(\mathrm{SO}_2(\mathbb{R})) = 1$. The measure $d^\times h$ on $F_q^{(1)}$ is chosen such that $\operatorname{vol}(F_q^{(1)}\cap\mathcal{O}_F) = 1$ for q finite and unramified, $\operatorname{vol}(F_q^{(1)}\cap\mathcal{O}_F) = 2$ for q finite and ramified and such that for $q = \infty$ it coincides with the measure dx/x on $K_\infty^{(1)} \simeq \mathbb{R}^\times$.

3.2.1 Local setup

To proceed further, it will be useful to set up some purely local notation. Fix a prime q and in what follows below we work entirely locally but without the subscript q. Suppose f_χ has conductor $\mathfrak{c}$ at q. By Casellman's theorem W_{f_χ} satisfies

$$W_{f_\chi}(gu) = \tilde{\omega}_\chi(u)W_{f_\chi}(g)$$

for $u \in \left\{\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} \in \mathrm{GL}_2(\mathfrak{r}), \gamma \equiv 0 \mod \mathfrak{c}\right\}$. Let F be a two dimensional algebra over $\mathbb{Q}_q$ (so F is either the split algebra $\mathbb{Q}_q\times\mathbb{Q}_q$ or a quadratic field extension of $\mathbb{Q}_q$) and denote by $\mathfrak{o}$ and $\mathfrak{r}$ the maximal orders in F and $\mathbb{Q}_q$ respectively. Fix an element $\xi \in F$ with $\operatorname{tr}(\xi) = 0$, so that $F = \mathbb{Q}_q + \mathbb{Q}_q\xi$. Also suppose we are given an embedding $j : F \hookrightarrow B_q$. Let $\tau \in F^\perp, \tau \neq 0$, where $F^\perp$ is the orthogonal complement to F in B for the norm form and the embedding given by j.

Define for $g \in \mathrm{GL}_2(\mathbb{Q}_q)$ and for $\varsigma_2 \in \mathcal{S}(F^\perp)$,

$$\text{(3.1)}\quad W_\eta(\varsigma_2,j,\tau,g) = |\operatorname{Nm}(\tau)|^{1/2}\int_{F^{(1)}} r_\psi\left(\begin{pmatrix}\operatorname{Nm}(\tau)^{-1} & 0\\ 0 & 1\end{pmatrix}g, hh_0\right)\varsigma_2(\tau)\eta(hh_0)d^\times h$$

where $h \in F^\times$ is any element with $\operatorname{Nm}(h) = \det(g)$. It is easy to check that if τ is replaced by $\beta\tau$, $\beta \in F^\times$, the integral (3.1) is multiplied by $\eta(\beta^i)^{-1}$. Since $\eta = (\chi^{-1}\chi_\nu)\circ N_{F/\mathbb{Q}_q}$,

$$W_\eta(\varsigma_2,j,g) := (\chi^{-1}\chi_\nu)(\operatorname{Nm}(\tau))\cdot W_\eta(\varsigma_2,j,\tau,g)$$

is independent of the choice of τ.

Suppose further that we are given a function $\varsigma \in \mathcal{S}(B)$. Write $\bar{\varsigma} = \sum_i \varsigma_{1,i} \otimes \varsigma_{2,i}$ and consider

$$(3.2)\quad I(\varsigma, j, \tau) = \sum_i \int_{\mathbb{Q}_q^\times} \int_K W_{f_\chi}(\mathbf{d}(a)k) W_\eta(\varsigma_{2,i}, j, \tau, \mathbf{d}(-a)k) \Phi^s(\varsigma_{1,i}, \mathbf{d}(-a)k) |a|^{-1} d^\times a dk$$

where $\varepsilon = \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Since $W_\eta(\varsigma_{2,i}, j, \tau, \cdot)\Phi^s_{1,i}(\cdot)$ is bilinear in $(\varsigma_{1,i}, \varsigma_{2,i})$, the expression on the right in (3.2) is independent of the decomposition $\bar{\varsigma} = \sum_i \varsigma_{1,i} \otimes \varsigma_{2,i}$. Clearly $I(\varsigma, j^u, \tau^u) = I(\varsigma, j, \tau)$ for $u \in U'$. If j and τ are fixed we will omit them from the notation.

For $\alpha, \beta \in \mathrm{GL}_2(\mathbb{Q}_q)$, set

$$(3.3)\quad J(\varsigma_1, \varsigma_2, \alpha, \beta) = \int_{\mathbb{Q}_q^\times} W_{f_\chi}(\mathbf{d}(a)\alpha) W_\eta(\varsigma_2, \mathbf{d}(-a)\beta) \Phi^s(\varsigma_1, \mathbf{d}(-a)\beta) d^\times a$$

and if $\varsigma = \sum_i \varsigma_{1,i} \otimes \varsigma_{2,i}$, then let

$$(3.4)\quad J(\varsigma, \alpha, \beta) = \sum_i J(\varsigma_{1,i}, \varsigma_{2,i}, \alpha, \beta).$$

4 Computing the local zeta integrals

We assume first that q is odd and split the computations we need into different cases as given by the table below:

	B	$\mathfrak{c}$	η	F	c
0	$M_2(\mathbb{Q}_q)$	1	*unram.*	*ram.*	1
IA	$M_2(\mathbb{Q}_q)$	q	*unram.*	*split*	1
IB	$M_2(\mathbb{Q}_q)$	q	*unram.*	*ram.*	1
II	$M_2(\mathbb{Q}_q)$	q^2	$\eta = (\eta_1, \eta_2)$, cond $(\eta_i) = q$	*split*	q
$IIIA$	$M_2(\mathbb{Q}_q)$	q	*ram.*	*inert*	q
$IIIB$	$M_2(\mathbb{Q}_q)$	q	$\eta = (\eta_1, \eta_2)$, cond $(\eta_i) = q$	*split*	q
IVA	B_q	q	*unram.*	*inert*	1
IVB	B_q	q	*unram.*	*ram.*	1

In what follows we use the following notation:

$$\mathbf{d}(a) = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}, \quad \bar{\mathbf{n}}(y) = \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}, \quad \underline{\mathbf{n}}(y) = \begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix}$$

$$\bar{\mathbf{m}}(a) = \begin{pmatrix} a & 1 \\ -1 & 0 \end{pmatrix}, \quad w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \varepsilon = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

Note that

$$(4.1)\quad \underline{\mathbf{n}}(y) = -w\bar{\mathbf{n}}(-y)w$$

$$(4.2)\quad \mathbf{d}(a)\bar{\mathbf{m}}(z) = \bar{\mathbf{n}}(-az)\mathbf{d}(a)w.$$

Recall also that the character η factors as $\eta = \tilde{\eta} \circ \mathrm{Nm}$ with $\tilde{\eta} = \chi^{-1}\chi_\nu$, a fact that we will use at times.

Note. If $F \simeq \mathbb{Q}_q \times \mathbb{Q}_q$ is split, then $\eta = (\eta_1, \eta_2)$ where $\eta_1 = \eta_2 = \tilde{\eta}$. However, for future use, we will often perform the computations below for an arbitrary character (η_1, η_2) with $\eta_1 \neq \eta_2$. It will be clear that the final formula obtained is valid even if $\eta_1 = \eta_2$ as is the case in our situation.

4.1 q odd, $\mathfrak{c}_{f_\chi} = (1)$

In this case, $\pi_{f_\chi} = \pi(\mu_1, \mu_2)$ is an unramified principal series and ς is invariant by $r_\psi(k, h)$ for $k \in K, \mathrm{Nm}(h) = \det(k)$.

4.1.1 Case 0: $B = M_2(\mathbb{Q}_q)$, η unramified, F ramified, $(c) = (1)$

Let $g \in F$ be such that $\mathrm{tr}(g) = 0$, $g^2 = \pi$, where π is a uniformizer in $\mathfrak{r}$. We may take $j_0(g) = \begin{pmatrix} 0 & 1 \\ \pi & 0 \end{pmatrix}$ and $\tau = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. Then $\bar{\varsigma} = \varsigma = \sum_{i=0}^{q-1} \mathbb{I}_{\mathfrak{o}+\frac{i}{q}g} \otimes \mathbb{I}_{(\mathfrak{o}-\frac{i}{q}g)\tau}$. Since $\Phi^s(\mathbb{I}_{\mathfrak{o}+\frac{i}{q}g}, \mathbf{d}(a)) = \delta_{0i}|a|^s$, and

$$W_\eta(\mathbb{I}_{\mathfrak{o}\tau}, \mathbf{d}(-a)) = \mu(F^{(1)})|a|^{1/2}\tilde{\eta}(a)\mathbb{I}_{\mathrm{Nm}(F^\times)}(a)$$

we have

$$I(\varsigma) = J(\varsigma, 1, 1) = (1 - \mu_1(q)\eta(g)q^{-s})^{-1}(1 - \mu_2(q)\eta(g)q^{-s})^{-1}.$$

4.2 q odd, $\mathfrak{c}_{f_\chi} = (q)$

In this case $q \mid N, \pi_{f_\chi} = \sigma(\mu_1, \mu_2)$ special, μ_i unramified with $\varsigma = \mathbb{I}_{\mathbb{O}_N \otimes \mathbb{Z}_q}$. Suppose $\mu_1 = |\cdot|^{\frac{1}{2}+it}, \mu_2 = |\cdot|^{-\frac{1}{2}+it}$. Then $\mathfrak{c} = (q)$ and

$$W_{f_\chi}(\mathbf{d}(a)) = |a|^{it}|a|\mathbb{I}_{\mathbb{Z}_q}(a), \qquad W_{f_\chi}(\mathbf{d}(a)w) = -|a|^{it}|aq|\mathbb{I}_{\mathbb{Z}_q}(aq).$$

We have $\hat{\varsigma} = \frac{1}{q}\mathbb{I}_{\hat{\mathbb{O}}_N}$ where $\hat{\mathbb{O}}_N = \begin{pmatrix} \mathfrak{r} & \mathfrak{r} \\ \frac{1}{q}\mathfrak{r} & \mathfrak{r} \end{pmatrix}$ if $B = M_2(\mathbb{Q}_q)$ and $\hat{\mathbb{O}}_N = u^{-1}R$ if B is ramified. It is easy to see using (4.1) that ς is invariant by $r_\psi(k, h)$ for any $k \in K_{0,q}, h \in F^\times \hookrightarrow B^\times, N(h) = \det(k)$. Now a set of representatives for $K_{0,q} \backslash K$ is $\{1, \bar{\mathbf{m}}(z), z = 0, \ldots, q-1\}$. Hence using (4.2) we see that

$$(q+1)I(\varsigma) = J(\varsigma, 1, 1) + qJ(\varsigma, w, w) = J(\varsigma, 1, 1) + \alpha_q q J(\hat{\varsigma}, w, 1).$$

4.2.1 Case IA: $B = M_2(\mathbb{Q}_q)$, η unramified, F is split, $d = 1$

Let g be such that $\mathrm{tr}(g) = 0$, $g^2 = u \in (\mathfrak{r}^\times)^2$, $\Lambda = \mathfrak{r} + \mathfrak{r}g$. Thus $s = 0, n = -u$. $\tilde{R} = \{\xi \in \mathfrak{r} \mid \xi^2 - u \equiv 0 \mod \pi\}$, $R = \{\alpha, -\alpha\}$ where $\alpha^2 = u$. For $\xi = \pm\alpha, j_\xi(g) = \begin{pmatrix} \xi & 1 \\ 0 & -\xi \end{pmatrix}$. Take $\tau = \begin{pmatrix} 1 & 0 \\ -2\xi & -1 \end{pmatrix}$ and let $\lambda_\xi : \mathbb{Q}_q \times \mathbb{Q}_q \simeq F, \lambda_\xi(\xi, -\xi) = g$. Then $\varsigma = \varsigma_1 \otimes \varsigma_2$, with

$\varsigma_1 = \mathbb{I}_{\mathfrak{o}}$ and $\varsigma_2 = \mathbb{I}_{\lambda_\xi(\mathfrak{r}\times\pi\mathfrak{r})\tau}$. $\hat{\varsigma} = \hat{\varsigma}_1 \otimes \hat{\varsigma}_2$ with $\hat{\varsigma}_1 = \mathbb{I}_{\mathfrak{o}}$ and $\hat{\varsigma}_2 = \frac{1}{q}\mathbb{I}_{\lambda_\xi(\mathfrak{r}\times\pi^{-1}\mathfrak{r})\tau}$ and $\Phi^s(\varsigma_1, \mathbf{d}(a)) = \Phi^s(\hat{\varsigma}_1, \mathbf{d}(a)) = |a|^s$. If $\eta \circ \lambda_\xi = (\eta_1, \eta_2)$, we have

$$\begin{aligned} W_\eta(\varsigma_2, \mathbf{d}(-a)) &= |a|^{1/2} \int_{\mathbb{Q}_q^\times} \varsigma_2((t, t^{-1}a)\tau)\eta(at^{-1}, t)d^\times t \\ &= |a|^{1/2}\mathbb{I}_{q\mathbb{Z}_q}(a)\eta_1(a) \sum_{i=0}^{v_q(a)-1} \left(\frac{\eta_2(q)}{\eta_1(q)}\right)^i \\ &= |a|^{1/2}\mathbb{I}_{q\mathbb{Z}_q}(a)\eta_1(q)\frac{\eta_2(a)-\eta_1(a)}{\eta_2(q)-\eta_1(q)} \end{aligned}$$

and likewise

$$W_\eta(\hat{\varsigma}_2, \mathbf{d}(-a)) = \frac{1}{q}|a|^{1/2}\mathbb{I}_{\frac{1}{q}\mathbb{Z}_q}(a)\eta_1(q)\frac{\eta_2(aq^2)-\eta_1(aq^2)}{\eta_2(q)-\eta_1(q)}.$$

Thus

$$J(\varsigma, 1, 1) = \eta_1(q)q^{-\frac{1}{2}-it-s}(1-\eta_1(q)|q|^{1/2+it+s})^{-1}(1-\eta_2(q)|q|^{1/2+it+s})^{-1}$$

$$J(\hat{\varsigma}, w, 1) = -\frac{1}{q}(\eta_1(q)q^{-1/2+it+s})(1-\eta_1(q)|q|^{1/2+it+s})^{-1}(1-\eta_2(q)|q|^{1/2+it+s})^{-1}$$

and

$$I(\varsigma) = -\frac{\eta_1(q)q^{-\frac{1}{2}+it+s}}{q+1}(1-q^{-2s})(1-\eta_1(q)|q|^{1/2+it+s})^{-1}(1-\eta_2(q)|q|^{1/2+it+s})^{-1}.$$

4.2.2 Case IB: $B = M_2(\mathbb{Q}_q)$, η unramified, F is ramified, $d = 1$

F is a ramified quadratic extension of $\mathbb{Q}_q$. Let g be such that $\mathrm{tr}(g) = 0$, $g^2 = \pi$, π a uniformizer in $\mathfrak{r}$, $\Lambda = \mathfrak{r} + \mathfrak{r}g$. Then $s = 0, n = -\pi, \tilde{R} = \{\xi \in \mathfrak{r} \mid \xi^2 - \pi \equiv 0 \mod \pi\} = \pi\mathfrak{r}$. $R = \{0\}, R' = \emptyset$. For $\xi = 0 \in R$, $j_0(g) = \begin{pmatrix} 0 & 1 \\ \pi & 0 \end{pmatrix}$. Take $\tau = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. Then $\varsigma = \varsigma_1 \otimes \varsigma_2$, where $\varsigma_1 = \mathbb{I}_{\mathfrak{o}}$ and $\varsigma_2 = \mathbb{I}_{\mathfrak{o}\tau}$. $\hat{\varsigma} = \hat{\varsigma}_1 \otimes \hat{\varsigma}_2$, where $\hat{\varsigma}_1 = q^{-1/2}\mathbb{I}_{\pi^{-1}\mathfrak{o}}, \hat{\varsigma}_2 = q^{-1/2}\mathbb{I}_{\pi^{-1}\mathfrak{o}\tau}$. Hence $\Phi^s(\varsigma_1, \mathbf{d}(a)) = q^{1/2}\Phi^s(\hat{\varsigma}_1, \mathbf{d}(a)) = |a|^s$,

$$W_\eta(\varsigma_2, \mathbf{d}(-a)) = |a|^{1/2}\eta(h_0)\int_{F^{(1)}} \varsigma_2(a(hh_0)^{-1}\tau)d^\times h = \frac{1}{2}\mu(F^{(1)})|a|^{1/2}\mathbb{I}_{\mathbb{Z}_q}(a)\eta(h_0)$$

$$W_\eta(\hat{\varsigma}_2, \mathbf{d}(-a)) = \frac{1}{2}\frac{1}{q^{1/2}}\mathrm{vol}(F^{(1)})|a|^{1/2}\mathbb{I}_{\mathbb{Z}_q}(aq)\eta(h_0)$$

for any $h_0 \in F$ with $\mathrm{Nm}(h_0) = a$. Let $\tilde{\pi}$ be a uniformizer in F. Then

$$\begin{aligned} J(\varsigma, 1, 1) &= \frac{1}{2}\mathrm{vol}(F^{(1)})(1-\eta(\tilde{\pi})|q|^{1/2+it+s})^{-1} \\ J(\hat{\varsigma}, w, 1) &= -\frac{1}{2}\mathrm{vol}(F^{(1)})\frac{\eta(\tilde{\pi})^{-1}}{q}\frac{|q|^{1/2-it-s}}{1-\eta(\tilde{\pi})|q|^{1/2+it+s}} \\ I(\varsigma) &= \frac{1}{2}\frac{\mathrm{vol}(F^{(1)})}{q+1}(1-\eta(\tilde{\pi})^{-1}|q|^{1/2-it-s})(1-\eta(\tilde{\pi})|q|^{1/2+it+s})^{-1}. \end{aligned}$$

4.2.3 Case IIIA,B: $B = \mathrm{M}_2(\mathbb{Q}_q)$, $d = q$

Let $g_0 \in F$, $\mathrm{tr}(g_0) = 0$, $g_0^2 = u \in \mathfrak{r}^\times$, $g = \pi g_0$. $\Lambda = \mathfrak{r} + \mathfrak{r}g$. Thus $s = 0, n = -\pi^2 u, \rho = 0$. $\tilde{R} = \{\xi \in \mathfrak{r} \mid \xi^2 - \pi^2 u \equiv 0 \mod \pi\}$, $R = R' = \{0\}$. For $\xi = 0 \in R$, $\varsigma_0(g) = \left(\begin{smallmatrix} 0 & 1 \\ \pi^2 u & 0 \end{smallmatrix}\right)$. Let $\tau = \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Then $\bar{\varsigma} = \varsigma = \sum_{i=0}^{q-1} \varsigma_{1,i} \otimes \varsigma_{2,i}$ where $\varsigma_{1,i} = \mathbb{I}_{\mathfrak{r}+(\pi\mathfrak{r}+i)g_0}$, $\varsigma_{2,i} = \mathbb{I}_{(\mathfrak{r}+(\pi\mathfrak{r}-i)g_0)\tau}$. Then $\hat{\varsigma} = \sum_i \hat{\varsigma}_{1,i} \otimes \hat{\varsigma}_{2,i}$ where $\hat{\varsigma}_{1,i}(x + yg_0) = \frac{1}{q}\psi(-iyu)\mathbb{I}_{\mathfrak{r}}(x)\mathbb{I}_{\frac{1}{\pi}\mathfrak{r}}(y)$, $\hat{\varsigma}_{2,i}((x + yg_0)\tau) = \frac{1}{q}\psi(-iyu)\mathbb{I}_{\mathfrak{r}}(x)\mathbb{I}_{\frac{1}{\pi}\mathfrak{r}}(y)$. Hence $\Phi^s(\varsigma_{1,i}, \mathbf{d}(a)) = \delta_{i0}|a|^s$, $\Phi^s(\hat{\varsigma}_{1,i}, \mathbf{d}(a)) = \frac{1}{q}|a|^s$. Now consider first the case IIIA, i.e. F is inert and η is ramified.

$$\begin{aligned} W_\eta(\varsigma_{2,0}, \mathbf{d}(-a)) &= |a|^{1/2}\eta(h_0)\int_{F^{(1)}} \varsigma_{2,0}(a(hh_0)^{-1}\tau)d^\times h \\ &= \begin{cases} \frac{2}{q+1}\tilde{\eta}(a), & \text{if } a \in (\mathbb{Z}_q^\times)^2 \\ |a|^{1/2}\tilde{\eta}(a)\mathbb{I}_{\mathbb{Z}_q}(a), & \text{if } v_q(a) \neq 0, v_q(a) \equiv 0 \mod 2 \end{cases} \end{aligned}$$

and is 0 otherwise. Since $\sum_i \hat{\varsigma}_{2,i}((x + yg_0)\tau) = \mathbb{I}_{\mathfrak{r}}(x)\mathbb{I}_{\mathfrak{r}}(y)$,

$$\sum_i W_\eta(\hat{\varsigma}_{2,i}, \mathbf{d}(-a)) = |a|^{1/2}\mathbb{I}_{\left\{\substack{v_q(\cdot)\geq 0 \\ v_q(a)\equiv 0 \mod 2}\right\}}(a)\tilde{\eta}(a)$$

we have

$$J(\varsigma, 1, 1) = \int_{\mathbb{Q}_q^\times} W_f(\mathbf{d}(a))W_\eta(\varsigma_{2,0}, \mathbf{d}(-a))|a|^{s-1}d^\times a = \begin{cases} \frac{1}{q+1} \text{ if } \tilde{\eta}|_{\mathbb{Z}_q^{\times 2}} = 1 \\ 0, \text{ otherwise} \end{cases}$$

$$\begin{aligned} J(\hat{\varsigma}, w, 1) &= \frac{1}{q}\int_{\mathbb{Q}_q^\times} W_f(\mathbf{d}(a))\Big\{\sum_i W_\eta(\hat{\varsigma}_{2,i}, \mathbf{d}(-a))\Big\}|a|^{s-1}d^\times a \\ &= -\frac{1}{q}\int_{\mathbb{Q}_q^\times} |a|^{it}|aq|\mathbb{I}_{\mathbb{Z}_q}(aq)\cdot|a|^{1/2}\tilde{\eta}(a)\mathbb{I}_{\mathbb{Z}_q}(a)\frac{(1+\epsilon_F(a))}{2}|a|^{s-1}d^\times a \\ &= 0 \end{aligned}$$

where ϵ_F is the quadratic character associated to F. Thus $I(\varsigma, \frac{1}{2}) = \frac{1}{(q+1)^2}$ if $\tilde{\eta}|_{\mathbb{Z}_q^{\times 2}} = 1$ and is zero otherwise.

Next consider the case IIIB, i.e. F is split and $\eta = (\eta_1, \eta_2)$ with cond $(\eta_i) = q$. Then taking $h = (t, t^{-1}), h_0 = (1, a), g_0 = (\xi, -\xi)$,

$$W_\eta(\varsigma_{2,0}, \mathbf{d}(-a)) = |a|^{1/2} \int_{\mathbb{Q}_q^\times} \varsigma_{2,0}((at^{-1}, t)\tau)\eta_1(t)\eta_2(at^{-1})d^\times t.$$

Now $\varsigma_{2,0}((at^{-1}, t)\tau) \neq 0 \Leftrightarrow at^{-1} + t \in \mathbb{Z}_q$ and $at^{-1} - t \in q\mathbb{Z}_q \Rightarrow 0 \leq v_q(t) \leq v_q(a)$. Suppose $v_q(a) = 0$. Then $\varsigma_{2,0}((at^{-1}, t)\tau) = 0$ unless $a \equiv \alpha^2 \mod q$ for some $\alpha \in \mathbb{Z}_q{}^\times$. In that case, $\varsigma_{2,0}((at^{-1}, t)\tau) = 1$ for $t \equiv \pm\alpha \mod q$ and $= 0$ otherwise. On the other hand, if $v_q(a) > 0$, then $\varsigma_{2,0}((at^{-1}, t)\tau) = 1$ if $0 < v_q(t) < v_q(a)$ and $= 0$ otherwise. For $a \in \mathbb{Z}_q^\times$, define $C(a) = \{(\eta_1\eta_2)(\alpha) + (\eta_1\eta_2)(-\alpha)\}/(q-1)$ if $a \equiv \alpha^2 \mod q$ and to be 0 if a is not a square mod q. Then

$$W_\eta(\varsigma_{2,0}, \mathbf{d}(-a)) = C(a)\mathbb{I}_{\mathbb{Z}_q^\times}(a) + \mathbb{I}_{q\mathbb{Z}_q}(a)C'(a)$$

where

$$C'(a) = \begin{cases} 0, \text{ if } \eta_1\eta_2^{-1} \text{ is ramified} \\ |a|^{1/2}\eta_2(a)\sum_{j=1}^{v_q(a)-1}(\eta_1\eta_2^{-1})(q)^j, \text{if } \eta_1\eta_2^{-1} \text{ is unramified.} \end{cases}$$

Since $\sum_i \hat{\varsigma}_{2,i}((x + yg_0)\tau) = \mathbb{I}_{\mathfrak{r}}(x)\mathbb{I}_{\mathfrak{r}}(y)$, setting $C''(a) = |a|^{1/2}\eta_2(a)\sum_{j=0}^{v_q(a)}(\eta_1\eta_2^{-1})(q)^j$,

$$\begin{aligned} \sum_i W_\eta(\hat{\varsigma}_{2,i}, \mathbf{d}(-a)) &= |a|^{1/2} \int_{0 \leq v_q(t) \leq v_q(a)} \eta_1(t)\eta_2(at^{-1})d^\times t \\ &= \begin{cases} 0, \text{ if } \eta_1\eta_2^{-1} \text{ is ramified, and} \\ C''(a), \text{ if } \eta_1\eta_2^{-1} \text{ is unramified.} \end{cases} \end{aligned}$$

Since η_2 is ramified, $\int_{\mathbb{Q}_q^\times} W_f(\mathbf{d}(a))C'(a)|a|^{s-1}d^\times a = \int_{\mathbb{Q}_q^\times} W_f(\mathbf{d}(a)w)C''(a)|a|^{s-1} = 0$. Thus $J(\hat{\varsigma}, w, 1) = 0$ and

$$\begin{aligned} I(\varsigma, s) &= \frac{1}{q+1}J(\varsigma, 1, 1) \\ &= \frac{1}{(q+1)(q-1)^2} \sum_{\alpha \in (\mathbb{Z}_q/q\mathbb{Z}_q)^\times} (\eta_1\eta_2)(\alpha) = \begin{cases} \frac{1}{q^2-1}, \text{ if } \eta_1\eta_2 \text{ is unramified} \\ 0, \text{ otherwise.} \end{cases} \end{aligned}$$

4.2.4 Case IV A,B: $B = B_q$, η unramified, F inert or ramified, $d = 1$

For $q \mid N^-$, let L_q be the unique unramified extension of $\mathbb{Q}_q$ of degree 2, π a uniformizer in $\mathbb{Z}_q$ and B_π be the quaternion algebra given by

$$\begin{aligned} B_\pi &= L_q + L_q u \\ um &= \overline{m}u \text{ for } m \in L \\ u^2 &= \pi. \end{aligned}$$

Fix an isomorphism $B \simeq B_\pi$. This isomorphism must necessarily identify $\mathcal{O}' \otimes \mathbb{Z}_q$ with $R_q + R_q u$, where R_q is the ring of integers of L_q. Also fix $\omega \in R_q$ with $\omega^2 = \alpha \in \mathbb{Z}_q^\times$, so that $R_q = \mathbb{Z}_q + \mathbb{Z}_q\omega$ and $R_q^0 = \mathbb{Z}_q\omega$.

If F is inert, $E_q = \{j_1, j_2\}$ where $j_i : F \to L_q$ are the two isomorphisms of F with L_q. Let $j = j_1$ or j_2 and take $\tau = u$. Then $\varsigma = \varsigma_1 \otimes \varsigma_2$, where $\varsigma_1 = \mathbb{I}_{R_q}$ and $\varsigma_2 = \mathbb{I}_{R_q u}$. $\hat{\varsigma} = \hat{\varsigma}_1 \otimes \hat{\varsigma}_2$, where $\hat{\varsigma}_1 = \mathbb{I}_{R_q}$ and $\hat{\varsigma}_2 = \frac{1}{q}\mathbb{I}_{\frac{1}{\pi}R_q u}$. Thus $\Phi^s(\varsigma_1, \mathbf{d}(a)) = \Phi^s(\hat{\varsigma}_1, \mathbf{d}(a)) = |a|^s$. If $h_0 \in F^\times$ is such that $N(h_0) = \pi^{-1}a$,

$$
\begin{aligned}
W_\eta(\varsigma_2, \tau, \mathbf{d}(-a)) &= |q|^{1/2}\int_{F^{(1)}} r_\psi(\mathbf{d}(\pi^{-1}a), hh_0)\varsigma_2(\tau)\eta(hh_0)d^\times h \\
&= |a|^{1/2}\eta(h_0)\int_{F^{(1)}} \varsigma_2(\pi^{-1}a(hh_0)^{-1}\tau)d^\times h \\
&= |a|^{1/2}\eta(h_0)\mathbb{I}_{q\mathbb{Z}_q, v_q(\cdot)\text{ odd}}(a) \\
W_\eta(\hat{\varsigma}_2, \tau, \mathbf{d}(-a)) &= \frac{1}{q}|a|^{1/2}\eta(h_0)\mathbb{I}_{\frac{1}{q}\mathbb{Z}_q, v_q(\cdot)\text{ odd}}(a) \\
J(\varsigma, 1, 1) &= \frac{|q|^{1/2+it+s}}{1-\eta(q)(|q|^{1/2+it+s})^2} \\
J(\hat{\varsigma}, w, 1) &= -\frac{1}{q}\frac{\eta(q)^{-1}|q|^{1/2-it-s}}{1-\eta(q)(|q|^{1/2+it+s})^2} \\
I(\varsigma) &= \frac{1}{q+1}\cdot\frac{|q|^{1/2+it+s}+\eta(q)^{-1}|q|^{1/2-it-s}}{1-\eta(q)(|q|^{1/2+it+s})^2}.
\end{aligned}
$$

If F is ramified, E_q consists of the class of a single element j. We may pick j and π such that $j(\delta) = u$ for δ a uniformizer in F. Let $\tau = \omega$. Then $\varsigma = \varsigma_1 \otimes \varsigma_2$, with $\varsigma_1 = \mathbb{I}_{\mathfrak{o}}$ and $\varsigma_2 = \mathbb{I}_{\mathfrak{o}\tau}$. $\hat{\varsigma} = \hat{\varsigma}_1 \otimes \hat{\varsigma}_2$, where $\hat{\varsigma}_1 = \frac{1}{\sqrt{q}}\mathbb{I}_{\mathfrak{q}^{-1}\mathfrak{o}}$ and $\hat{\varsigma}_2 = \frac{1}{\sqrt{q}}\mathbb{I}_{\mathfrak{q}^{-1}\mathfrak{o}\tau}$, $\mathfrak{q}$ being the prime ideal of F over q. Thus $\Phi^s(\varsigma_1, \mathbf{d}(a)) = \sqrt{q}\Phi^s(\hat{\varsigma}_1, \mathbf{d}(a)) = |a|^s$. If $h_0 \in F^\times$ is such that $N(h_0) = \alpha^{-1}a$,

$$
\begin{aligned}
W_\eta(\varsigma_2, \tau, \mathbf{d}(-a)) &= \int_{F^{(1)}} r_\psi(\mathbf{d}(\alpha^{-1}a), hh_0)\varsigma_2(\tau)\eta(hh_0)d^\times h \\
&= |a|^{1/2}\eta(h_0)\int_{F^{(1)}} \varsigma_2(\alpha^{-1}a(hh_0)^{-1}\tau)d^\times h \\
&= \operatorname{vol}(F^{(1)})|a|^{1/2}\eta(h_0)\mathbb{I}_{\mathbb{Z}_q}(a)\mathbb{I}_{N(F^\times)}(\alpha^{-1}a) \\
W_\eta(\hat{\varsigma}_2, \tau, \mathbf{d}(-a)) &= \operatorname{vol}(F^{(1)})\frac{1}{\sqrt{q}}|a|^{1/2}\eta(h_0)\mathbb{I}_{\frac{1}{q}\mathbb{Z}_q}(a)\mathbb{I}_{N(F^\times)}(\alpha^{-1}a) \\
J(\varsigma, 1, 1) &= \frac{1}{2}\operatorname{vol}(F^{(1)})\cdot\frac{1}{1-\eta(\delta)|q|^{1/2+it+s}} \\
J(\varsigma, w, 1) &= -\frac{\operatorname{vol}(F^{(1)})}{2q}|q|^{1/2-it-s}\eta(\delta)^{-1}\frac{1}{1-\eta(\delta)|q|^{1/2+it+s}} \\
I(\varsigma) &= \frac{\operatorname{vol}(F^{(1)})}{2(q+1)}\frac{1+\eta(\delta)^{-1}|q|^{1/2-it-s}}{1-\eta(\delta)|q|^{1/2+it+s}}.
\end{aligned}
$$

4.3 q odd, $\mathfrak{c}_{f_\chi} = (q^2)$

We will only consider the case when B is split since this will be sufficient for our purposes. Then we are in Case II i.e. $B = \mathrm{M}_2(\mathbb{Q}_q)$, F is split, $\eta = (\eta_1, \eta_2)$, $\mathrm{cond}\,(\eta_i) = q$, $d = q$, $\eta_i\chi$ is unramified. By the analysis of Lemma 2.4 and the case IA, we see that

$$I(\varsigma) = -\frac{(q+\chi_q(-1))^2}{q(q-1)^2} I'$$

where

$$I' = \frac{(\eta_1\chi)(q) q^{-\frac{1}{2}+it+s}}{q+1}(1-q^{-2s}) \\ \times (1-(\eta_1\chi)(q)|q|^{1/2+it+s})^{-1}(1-(\eta_2\chi)(q)|q|^{1/2+it+s})^{-1}.$$

4.4 $q = 2$

In this case, $\pi_{f_\chi} = \pi(\mu_1, \mu_2)$ is an unramified principal series representation, ς is right invariant by $r_\psi(k, h)$ for $k \in K, \mathrm{Nm}(h) = \det k$. For simplicity, we will assume that $\xi_0 \equiv 4 \mod 8$ and later deduce the main formula in the general case from the knowledge of the formula in this case. Let $g = \sqrt{\xi'}$, $j_0(g) = \begin{pmatrix} 0 & 1 \\ \xi' & 0 \end{pmatrix}$, $\tau = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. Then $\bar{\varsigma} = \varsigma = \sum_{i=0}^{1}\sum_{j=0}^{1} \varsigma_1^{ij} \otimes \varsigma_2^{ij}$, where $\varsigma_1^{ij} = \mathbb{I}_{\frac{i}{2}+\frac{j}{2}g+\mathfrak{o}}$, $\varsigma_2^{ij} = \mathbb{I}_{(-\frac{i}{2}-\frac{j}{2}g+\mathfrak{o})\tau}$. Now $\Phi^s(\varsigma_1^{ij}, \mathbf{d}(a)) = \delta_{i0}\delta_{j0}|a|^s$, and

$$W_\eta(\mathbb{I}_{\mathfrak{o}\tau}, \mathbf{d}(-a)) = |a|^{1/2}\tilde{\eta}(a)\mathrm{vol}(F^{(1)})\mathbb{I}_{\mathbb{Z}_2 \cap \mathrm{Nm}(F^\times)}(a)$$

if $\eta = \tilde{\eta} \circ \mathrm{Nm}$. Thus for π any uniformizer in $\mathbb{Z}_2 \cap \mathrm{Nm}(F^\times)$, we have

$$I(\varsigma) = J(\varsigma, 1, 1) = \frac{1}{\mu_1(\pi) - \mu_2(\pi)} \sum_{n=0}^{\infty} (\mu_1(\pi^{n+1})\tilde{\eta}(\pi^n) - \mu_1(\pi^{n+1})\tilde{\eta}(\pi^n))|\pi^n|^s \\ = (1-\mu_1(\pi)\tilde{\eta}(\pi)2^{-s})^{-1}(1-\mu_2(\pi)\tilde{\eta}(\pi)2^{-s})^{-1}.$$

5 The final formula

Proof of Thm. 1.1. We only need to put together our computations. We will assume that ξ_0 is not divisible by any primes that divide ν but do not divide N and that $\xi_0 \equiv 4 \mod 8$. The formula in the general case can easily be deduced from this special case using the main result of [3].

Case (a): In this case, it follows from the results in [5] that $I(\varsigma) = L_q(\frac{1}{2}, \pi \otimes \chi_\nu)L_q(\frac{1}{2}, \pi \otimes \chi_{\xi_0})L_q(1, \epsilon_F)^{-1}$. Set $C_q = 1$.

Case (b): We have assumed ξ is prime to ν, hence we are in Case 0. Since $\mu_1(q)\eta(g) = \mu_1(q) \cdot \chi^{-1}\chi_\nu(-\nu\xi_0) = (\mu_1\chi^{-1})(q) \cdot \chi_\nu(q)$, we have $I(\varsigma) = L_q(s, \pi \otimes \chi_\nu)L_q(s, \pi \otimes \chi_{\xi_0})L_q(1, \epsilon_F)^{-1}$. Set $C_q = 1$.

Case (c1): If $q \nmid \xi$, F must be split and we are in Case IA: $I(\varsigma,\frac{1}{2}) = \frac{\eta_1(q)w_q(f_\chi)}{q+1}L_q(\frac{1}{2},\pi\otimes\chi_\nu)L_q(\frac{1}{2},\pi\otimes\chi_{\xi_0})L_q(1,\epsilon_F)^{-1}$. Set $C_q = 4(q+1)^{-2}$. If $q \mid \xi$, we are in Case IB: since $\eta_q(\tilde{\pi}) = \chi_q^{-1}\chi_{\nu,q}(-\nu\xi_0) = \chi_q^{-1}\chi_{\nu,q}(q)$, $|q|^{it} = -\chi_q(q)w_q$ and $\chi_{\nu,q}(q) = w_q$, $I(\varsigma,\frac{1}{2}) = \frac{1}{q+1}L_q(\frac{1}{2},\pi\otimes\chi_\nu)L_q(\frac{1}{2},\pi\otimes\chi_{\xi_0})L_q(1,\epsilon_F)^{-1}$. Set $C_q = (q+1)^{-2}$ in this case.

Case (c2): We have $q \nmid \xi$. Since $(\nu\xi_0,q)_q = (\nu,q)_q(\xi_0,q)_q = (\nu,q)_q\chi_{0,q}(-1)w_q = 1$, we see that F must be split at q and we are in Case II. Then $I(\varsigma,\frac{1}{2}) = \frac{q}{(q-1)^2(q+1)}(1+\chi_q(-1)q^{-1})^2L_q(\frac{1}{2},\pi\otimes\chi_\nu)L_q(\frac{1}{2},\pi\otimes\chi_{\xi_0})L_q(1,\epsilon_F)^{-1}$. Set $C_q = 4(1+\chi_q(-1)q^{-1})^2/q^2(q+1)^2$.

Case (c3): If $q \nmid \xi$, F is ramified and we are in case IB: we may take $\tilde{\pi} = \sqrt{\nu\xi_0}$. Then $\eta(\tilde{\pi})|q|^{it} = (\nu,-\nu\xi_0)_q(-w_q) = -(\nu,\xi_0)_q w_q = -(q,\xi_0)_q w_q = -1$ since $\left(\frac{\xi_0}{q}\right) \neq -w_q$. $I(\varsigma,\frac{1}{2}) = \frac{1}{q+1}L_q(\frac{1}{2},\pi\otimes\chi_\nu)L_q(\frac{1}{2},\pi\otimes\chi_{\xi_0})L_q(1,\epsilon_F)^{-1}$. Set $C_q = (q+1)^{-2}$.

If on the other hand, $q \mid \xi$, F is either split or inert. If F is split, we are in Case IIIB: since $\eta_{1,q} = \eta_{2,q} = \chi_q^{-1}\chi_{\nu,q}$, $\eta_{1,q}\eta_{2,q}$ is unramified and $I(\varsigma,\frac{1}{2}) = \frac{1}{q^2-1} = \frac{q}{(q+1)(q-1)^2}L_q(\frac{1}{2},\pi\otimes\chi_\nu)L_q(\frac{1}{2},\pi\otimes\chi_{\xi_0})L_q(1,\epsilon_F)^{-1}$. Set $C_q = (q+1)^{-2}$ in this case. If F is inert, we are in case IIIA. Since $\tilde{\eta} = \chi^{-1}\chi_\nu$, $I(\varsigma,\frac{1}{2}) = \frac{1}{(q+1)^2} = \frac{q}{(q+1)^3}L_q(\frac{1}{2},\pi\otimes\chi_\nu)L_q(\frac{1}{2},\pi\otimes\chi_{\xi_0})L_q(1,\epsilon_F)^{-1}$. Set $C_q = (q+1)^{-2}$.

Case (d1): Suppose that χ is unramified at q. If $q \nmid \xi$, F must be inert at q and we are in Case IVA: $I(\varsigma,\frac{1}{2}) = -\frac{1}{q+1}\chi_q(q)w_qL_q(\frac{1}{2},\pi\otimes\chi_\nu)L_q(\frac{1}{2},\pi\otimes\chi_{\xi_0})L_q(2s,\epsilon_F)^{-1}$. Set $C_q = 4/(q^2-1)$. If $q \mid \xi$, F is ramified at q and we are in Case IVB: since $\eta(\delta)^{-1}|q|^{-it} = -(\nu,q)_q w_q = 1$, $I(\varsigma,\frac{1}{2}) = \frac{1}{(q+1)}L_q(\frac{1}{2},\pi\otimes\chi_\nu)L_q(\frac{1}{2},\pi\otimes\chi_{\xi_0})L_q(1,\epsilon_F)^{-1}$. Set $C_q = (q^2-1)^{-1}$.

Case (e): In this case, it follows from the computation above that $I(\varsigma) = L_q(\frac{1}{2},\pi\otimes\chi_\nu)L_q(\frac{1}{2},\pi\otimes\chi_{\xi_0})L_q(1,\epsilon_F)^{-1}$. Set $C_q = 1$.

Case (f): Set $C_\infty = 2\cdot\Gamma(k)^2$ (see [5] Prop. 4.2.6 and Sec. 5.3 for this case).

Let us now assume first that χ is unramified at N^-. Then we see from the computations above that

$$|a_\xi(h)|^2 = C(f,\chi,\nu)\pi^{-2k}|\nu\xi|^{k-\frac{1}{2}}L(\frac{1}{2},\pi_f\otimes\chi_\nu)L(\frac{1}{2},\pi_f\otimes\chi_{\xi_0})\cdot\frac{\langle g_\chi,g_\chi\rangle}{\langle f_\chi,f_\chi\rangle}.$$

Suppose now that χ is possibly ramified at N^-. Then χ decomposes canonically as $\chi = \chi^+\chi^-$ where χ^+ (resp. χ^-) has conductor dividing N^+ (resp. N^-.) Since g_χ is a scalar multiple of $g_{\chi^+}\cdot\chi^-\circ\mathrm{Nm}$, we have $g_\chi = \alpha\cdot g_{\chi^+}\cdot\chi^-\circ\mathrm{Nm}$, say. Now, by what we have just seen

$$|a_\xi(h)|^2 = |\alpha|^2C(f,\chi^+,\nu)\pi^{-2k}|\nu\xi|^{k-\frac{1}{2}}L(\frac{1}{2},\pi_f\otimes\chi_\nu)L(\frac{1}{2},\pi_f\otimes\chi_{\xi_0})\cdot\frac{\langle g_{\chi^+},g_{\chi^+}\rangle}{\langle f_{\chi^+},f_{\chi^+}\rangle}.$$

So we only need to relate $\frac{\langle g_{\chi^+}, g_{\chi^+}\rangle}{\langle f_{\chi^+}, f_{\chi^+}\rangle}$ to $\frac{\langle g_\chi, g_\chi\rangle}{\langle f_\chi, f_\chi\rangle}$. Clearly, $|\alpha|^2\langle g_{\chi^+}, g_{\chi^+}\rangle = \langle g_\chi, g_\chi\rangle$. Also by a familiar computation

$$\frac{\langle f_{\chi^+}, f_{\chi^+}\rangle}{\langle f_\chi, f_\chi\rangle} = \prod_{q|\mathfrak{c}_{\chi^-}} L_q(1, Ad^0(f)) = \prod_{q|\mathfrak{c}_{\chi^-}} \frac{1}{1-\frac{1}{q}} = \prod_{q|\mathfrak{c}_{\chi^-}} \frac{q}{q-1}.$$

Now we define C_q for the remaining cases that we did not consider previously.

Case (d1) with χ_q ramified: In this case, $q \nmid \xi$ and F is inert. Thus set $C_q = \frac{4q}{(q-1)^2(q+1)}$.

Case (d2): In this case, χ_q is ramified, $q \nmid \xi$, F is ramified at q. Set $C_q = \frac{q}{(q-1)^2(q+1)}$.

Finally, if $\xi_0 \not\equiv 0 \mod 4$ or ξ_0, we still define C_q for q in cases (b) and (e) to be 1. The formula of Thm. 1.1 follows now from the computations above, those in [5], Thm. 10.1 of [3], Prop. 2.4 and the discussion in Sec. 3.2. □

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