Math 494 Winter 2024 HW1

1. **Product Rings.** If R and S are two rings, we define

$$R \times S = \{(x, y) : x \in R, y \in S\}.$$

   (a) Show that $R \times S$ is naturally a ring. What is the additive identity? Multiplicative identity?

   (b) Consider the projection maps:

$$p_1 : R \times S \to R, \quad (x, y) \mapsto x$$
$$p_2 : R \times S \to S, \quad (x, y) \mapsto y$$

   Are these maps ring homomorphisms?

   (c) Consider the inclusion maps

$$i_1 : R \to R \times S, \quad x \mapsto (x, 0)$$
$$i_2 : S \to R \times S, \quad y \mapsto (0, y)$$

   Are these maps ring homomorphisms? Are they group homomorphisms of the underlying additive groups?

   (d) If R_1 is a subring of R and S_1 is a subring of S, show that $R_1 \times S_1$ is a subring of $R \times S$.

   (e) * If I is an ideal in R and J is an ideal in S, show that $I \times J$ is an ideal in $R \times S$. Is every ideal in $R \times S$ of this form?

2. **Operations on ideals.** Let I, J be ideals in a ring R. Then we can define ideals $I + J$, $I \cap J$, IJ.

$$I + J = \{a + b : a \in I, b \in J\},$$

   while IJ is the ideal *generated by* products ab, where $a \in I$ and $b \in J$. Note that it may also be described as the set of finite sums

$$\sum_i a_i b_i$$

   where $a_i \in I$ and $b_i \in J$.

   (a) Show that $IJ \subseteq I \cap J \subseteq I, J \subseteq I + J$.

   (b) If $I = m\mathbf{Z}$ and $J = n\mathbf{Z}$ in $R = \mathbf{Z}$, identify the ideals $I + J$, $I \cap J$, IJ explicitly.

   (c) Show that

$$(I + J)(I \cap J) \subset IJ.$$

   (d) * We say two ideals I and J are coprime if $I + J = (1)$. Show that if I and J are coprime then $IJ = I \cap J$. What familiar statement is this in the ring $\mathbf{Z}$?

3. **Nilpotents and the radical of an ideal.** An element x in R is said to be nilpotent if there exists a positive integer n such that $x^n = 0$.

   (a) Show that the set of nilpotent elements in R is an ideal in R. This is called the *nilradical* of R, denoted $\mathfrak{N}(R)$ or just $\mathfrak{N}$ if R is known from context.

(b) More generally, if I is any ideal in R, we define

$$r(I) = \{x \in R : x^n \in I \text{ for some positive integer } n\}.$$

Show that $r(I)$ is an ideal in R containing I and that $r(r(I)) = r(I)$. (Remark: another common notation for $r(I)$ is $\sqrt{I}$.)

(c) * Show that if I and J are ideals in R, then

$$r(I \cap J) = r(IJ) = r(I) \cap r(J)$$

and

$$r(I + J) = r(r(I) + r(J)).$$

(d) Let k be a field and let $I = (X^2, XY)$ in the ring $R = k[X, Y]$. Is I a principal ideal? Is $r(I)$ a principal ideal?

4. **Polynomial rings.** Let R be a ring and $R[X]$ the polynomial ring in a variable X over R. Let

$$f = a_0 + a_1 X + \cdots + a_n X^n \in R[X]$$

with $a_i \in R$.

(a) * Show that f is a unit in $R[X]$ if and only if a_0 is a unit in R and $a_1, \ldots, a_n$ are nilpotent in R.

(b) * Show that f is nilpotent if and only if $a_0, a_1, \ldots, a_n$ are nilpotent in R.

(c) Show that f is a zero-divisor if and only if there exists some nonzero element $a \in R$ such that $a \cdot a_i = 0$ for all i.

(d) We say f is primitive if and only if $(a_0, \ldots, a_n) = (1)$ in R. Show that if $f, g \in R[X]$, then fg is primitive if and only if f and g are primitive.

5. **Power series rings (15 pts).** Let R be a ring. We define

$$R[[X]] = \{a_0 + a_1 X + a_2 X^2 + \cdots \;\; ; a_i \in R\}.$$

This is the ring of *formal power series* with coefficients in R in the variable X. Addition and multiplication are defined in the obvious way. Let

$$f = a_0 + a_1 X + a_2 X^2 + \cdots \in R[[X]].$$

(a) * Show that f is a unit in $R[[X]]$ if and only if a_0 is a unit in R.

(b) * Show that if f is nilpotent, then each a_i is nilpotent in R. Do you think the converse to this is true?

(c) * A ring R is said to be Noetherian if every ideal in R is finitely generated. Show that if R is Noetherian then the converse to the previous subpart is true.

6. **Rings of continuous functions.** Let R be the ring of continuous functions $f : \mathbf{R} \to \mathbf{R}$. (Here $\mathbf{R}$ is the usual real line.) Note that addition and multiplication are defined pointwise, so that

$$(f + g)(x) = f(x) + g(x)$$
$$(fg)(x) = f(x)g(x)$$

The additive identity is the constant function 0 while the multiplicative identity is the constant function 1.

(a) Explicitly describe the units in R.

(b) If $T \subset \mathbf{R}$, show that the set of functions $\mathcal{I}(T)$ in R that vanish on T is an ideal.

(c) If $T \subset T'$, what is the relation between $\mathcal{I}(T)$ and $\mathcal{I}(T')$?

(d) If $\mathcal{I}(T) \subset \mathcal{I}(T')$, what is the relation between T and T'?

(e) * Let $I = (\sin(x))$ and $J = (\cos(x))$ be the principal ideals generated by the functions $\sin(x)$ and $\cos(x)$ respectively. Is it true that $IJ = I \cap J$?

(f) * Is R a noetherian ring?