Math 494 Winter 2024 HW2

Please submit only the starred subparts (11 in all). 10 points for each subpart, except the two subparts in Problem 4 will be worth 5 each. 100 points total.

1. **Homomorphisms and ideals.**

   (a) Artin Ex. 11.3.3. Find generators for the kernels of the following maps:

   i. $\mathbf{R}[X, Y] \to \mathbf{R}$ defined by $f(X, Y) \mapsto f(0, 0)$.

   ii. $\mathbf{R}[X] \to \mathbf{C}$ defined by $f(X) \mapsto f(2 + i)$.

   iii. $\mathbf{Z}[X] \to \mathbf{R}$ defined by $f(X) \mapsto f(1 + \sqrt{2})$.

   iv. $\mathbf{Z}[X] \to \mathbf{C}$ defined by $X \mapsto \sqrt{2} + \sqrt{3}$.

   v. * $\mathbf{C}[X, Y, Z] \to \mathbf{C}[t]$ defined by $X \mapsto t$, $Y \mapsto t^2$, $Z \mapsto t^3$.

   (b) Find all the ideals in the power series ring $k[[X]]$, where k is a field.

2. **Extensions and contractions of ideals.** Let $\varphi : A \to B$ be a homomorphism of rings. For J an ideal of B, we define J^c (the contraction of J) to be the $\varphi^{-1}(J)$. For I an ideal of A, we define I^e (the extension of I) to be the ideal of B generated by $\varphi(I)$.

   (a) Show that $I \subseteq I^{ec}$ and that in general this inclusion is strict. Show that $J^{ce} \subseteq J$ and that in general this inclusion is strict.

   (b) * Show that $I^{ece} = I^e$ and $J^{cec} = J^c$.

   (c) * Show that if $\mathfrak{q}$ is a prime ideal of B, then $\mathfrak{q}^c$ is a prime ideal of A. Show that this is not true if the word *prime* is replaced by *maximal*. (It was one of Grothendieck's great insights that this property of prime ideals makes them much better to work with than maximal ideals.)

   Remark: On the other hand, if $\mathfrak{p}$ is a prime ideal of A, the ideal $\mathfrak{p}^e$ is typically not prime. Deciding whether or not $\mathfrak{p}^e$ is prime can be a very subtle problem. and is tied to deep arithmetic questions. See for example Ex. 3(c) and 3(d) below.

3. **Chinese remainder theorem.**

   (a) State and prove the analog of the Chinese remainder theorem for n ideals $I_1, \ldots, I_n$.

   (b) * Find all the ideals in the ring $\mathbf{Z}[i]/(5)$. (You should write down explicit generators for the ideals.) Which ones of these are prime? Which are maximal?

4. **Quotient rings and some arithmetic.**

   (a) Artin Ex 4.3. Identify the following rings:

   i. $\mathbf{Z}[X]/(X^2 - 3, 2X + 4)$.

   ii. $\mathbf{Z}[i]/(2 + i)$

   iii. $\mathbf{Z}[X]/(6, 2X - 1)$

   iv. * $\mathbf{Z}[X]/(2X^2 - 4, 4X - 5)$

   v. $\mathbf{Z}[X]/(X^2 + 3, 5)$.

   (b) * Artin Ex. 4.4. Are the rings $\mathbf{Z}[X]/(X^2 + 7)$ and $\mathbf{Z}[X]/(2X^2 + 7)$ isomorphic?

(c) * Let $R = \mathbf{Z}[i]$, the ring of Gaussian integers and let $\mathbf{F}_p \simeq \mathbf{Z}/p\mathbf{Z}$ denote the finite field with p elements. Analyze the structure of the ring $R/(p)$, where p is a prime in $\mathbf{Z}$. Show that:

$$R/(p) \simeq \begin{cases} \mathbf{F}_2[t]/(t^2), \text{ if } p = 2; \\ \mathbf{F}_p \times \mathbf{F}_p, \text{ if } p \text{ is odd and } -1 \text{ is a square mod } p; \\ \text{ a field with } p^2 \text{ elements, if } p \text{ is odd and } -1 \text{ is a non-square mod } p. \end{cases}$$

For the first few odd primes p, decide which case of the above occurs. Make a table and decide if there is some visible pattern. (You can do this by hand, or better, write some code to do it quickly.)

(d) * Now formulate and work out the analog of the previous subpart for the ring $\mathbf{Z}[\omega]$, where ω is a primitive cube root of unity.

5. **The spectrum of a ring.** It turns out that to every ring A one can attach a topological space $X = \mathrm{Spec}(A)$ defined as follows. The points of $X = \mathrm{Spec}(A)$ are the prime ideals of A. For any subset E of A, we define $V(E)$ to be the set of all prime ideals of A containing E.

   (a) * Show that the sets $V(E)$ satisfy the axioms defining the *closed sets* for some topology on X. This topology is called the *Zariski topology* on X.

   (b) Show that if I is the ideal generated by E, then $V(E) = V(I)$.

   (c) For any element $f \in A$, define X_f to be the complement of $V(f)$ in X. Show that the subsets V_f form a basis for the open sets in the Zariski topology. (This means that given any open set U and an element $x \in U$, there exists an X_f such that $x \in X_f \subset U$.)

   (d) For a homomorphism $\varphi : A \to B$, one gets a map (in the opposite direction!)

   $$\varphi^* : \mathrm{Spec}(B) \to \mathrm{Spec}(A),$$

   defined by

   $$\varphi^*(\mathfrak{q}) = \varphi^{-1}(\mathfrak{q}) = \mathfrak{q}^c.$$

   Show that the map φ^* is continuous for the Zariski topologies on $\mathrm{Spec}(A)$ and $\mathrm{Spec}(B)$.

   (e) * Draw pictures (!) of $\mathrm{Spec}(\mathbf{Z})$, $\mathrm{Spec}(\mathbf{R})$, $\mathrm{Spec}(\mathbf{C}[X])$, $\mathrm{Spec}(\mathbf{R}[X])$.

   (f) Draw pictures of $\mathrm{Spec}(\mathbf{Z}[i]/(3))$, $\mathrm{Spec}(\mathbf{Z}[i]/(5))$, $\mathrm{Spec}(\mathbf{Z}[i]/(2))$.

   (g) * Draw a picture of the map

   $$\mathrm{Spec}(\mathbf{Z}[i]) \to \mathrm{Spec}(\mathbf{Z}).$$