Math 494 Winter 2024 HW3

Please submit only the starred subparts (11 in all). 10 points for each subpart, except subparts (b) and (c) of Problem 3 will be worth 5 each. 100 points total.

1. **Localization**

   (a) * Let A be a ring and $f \in A$. We often write A_f for the localization of A at the multiplicative set $\{1, f, f^2, \ldots\}$ generated by f. Show that there is a canonical isomorphism
   $$\frac{A[X]}{fX-1} \simeq A_f.$$
   Using this, identify the rings $\mathbf{Z}[X]/(2X-1)$ and $(\mathbf{Z}/6\mathbf{Z})[X]/(2X-1)$ as more familiar rings.
   Remark: the notation A_f works well when A is a general ring. In particular cases, it may not be appropriate. eg. if $A = \mathbf{Z}$ and $f = p$, we do not use this notation, since the symbol $\mathbf{Z}_p$ is reserved for a different ring, namely the ring of p-adic integers.

   (b) * Let A be a ring and let S be a multiplicative subset, all whose elements are units in A. Show that the natural map $A \to S^{-1}(A)$ is an isomorphism. In particular, if K is a field, then there is a canonical isomorphism
   $$\mathrm{Frac}(K) \simeq K.$$
   Deduce that if R is an integral domain, then $\mathrm{Frac}(\mathrm{Frac}(R)) \simeq \mathrm{Frac}(R)$.

   (c) Let A be a ring and let S and T be multiplicative subsets of A. (Note that the set ST of products of elements of S and T is also a multiplicative subset.) Let $\overline{T}$ denote the image of T in $S^{-1}A$. (Note that $\overline{T}$ is a multiplicative subset of $S^{-1}(A)$. Show that
   $$\overline{T}^{-1}(S^{-1}A) \simeq (ST)^{-1}(A).$$
   What is the intuitive meaning of this result?

   (d) Let A be a ring and let S_0 denote the set of non-zero-divisors in A. This is a multiplicative subset of A. The localization $S_0^{-1}(A)$ is called the total ring of fractions of A. Show that
      i. The map $A \to S_0^{-1}(A)$ is injective, and S_0 is the largest multiplicative subset of A for which this is true.
      ii. Every element of $S_0^{-1}(A)$ is either a zero-divisor or a unit.
      iii. If every element of A is either a zero-divisor or a unit, then the map $A \to S_0^{-1}(A)$ is an isomorphism. (Note that this generalizes part 1b above.)

   (e) * Classify all possible localizations of the ring $\mathbf{Z}/60\mathbf{Z}$.

   (f) * Classify the prime ideals in the ring $\mathbf{Z}[X]$. (This is tricky; feel free to ask for a hint.)

   (g) Draw a picture of $\mathrm{Spec}(\mathbf{Z}[X])$.

   (h) Let $\varphi : A \to B$ be a ring homomorphism. Suppose that $\mathfrak{p}$ is a prime ideal in A that is the contraction of some ideal of B. Show that there exists a *prime* ideal $\mathfrak{q}$ of B such that $\mathfrak{q}^c = \mathfrak{p}$.

   (i) * A local ring is a ring with a unique maximal ideal. Let M be a smooth manifold (you can think of just $\mathbf{R}^n$ or $\mathbf{R}$ if you don't know what that means) and x a point on M. We define a germ of a function at x to be an equivalence class of a pair (U, f), where U is an open set of M containing x and $f : U \to \mathbf{R}$ is a smooth (i.e., C^∞) function on U, with two such pairs (U_1, f_1) and (U_2, f_2) considered equivalent if there exists an open subset V contained in $U_1 \cap U_2$ with $f_1|_V = f_2|_V$. Show that the set R_x of germs of functions at x is naturally a ring, and moreover that it is a local ring.

(j) * Let A be a ring. In the previous homework, we defined the space $\operatorname{Spec}(A)$ equipped with its Zariski topology. If you're only familiar with spaces such as Euclidean spaces, the Zariski topology may appear a bit pathological. For example, $\operatorname{Spec}(A)$ is rarely Hausdorff. Show that $\operatorname{Spec}(A)$ is Hausdorff if and only if every prime ideal of A is maximal.

2. **Prime ideals**

Let A be a ring.

(a) * Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ be prime ideals of A and let I be an ideal in A. Suppose

$$I \subseteq \mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_n.$$

Show that $I \subseteq \mathfrak{p}_i$ for some i.

(b) Let $I_1, \ldots, I_n$ be ideals in A and let $\mathfrak{p}$ be a prime ideal in A. Suppose that

$$\mathfrak{p} \supseteq I_1 \cap \cdots \cap I_n.$$

Show that $\mathfrak{p} \supseteq I_j$ for some j. In particular, if $\mathfrak{p} = I_1 \cap \cdots \cap I_n$, then $\mathfrak{p} = I_j$ for some j.

(c) * Show that the intersection of all prime ideals of A is equal to the nilradical of A. More generally, if I is an ideal in A, then the intersection of all prime ideals of A containing I equals $r(I)$, the radical of I.

3. **The spectrum of a ring, contd.**

Let A be a ring, and let $X = \operatorname{Spec}(A)$. We study various properties of the *basic open sets* X_f for $f \in A$. Show that:

(a) $X_f \cap X_g = X_{fg}$.

(b) * $X_f = X$ iff f is a unit.

(c) * $X_f = \emptyset$ iff f is nilpotent.

(d) $X_f = X_g$ iff $r((f)) = r((g))$.

(e) * Show that X is *quasi-compact*, namely every open cover of X has a finite subcover. (Since some texts require that compact topological spaces be Hausdorff, and $\operatorname{Spec}(A)$ is rarely Hausdorff, it is typical to use the term quasi-compact for the property that every open cover have a finite subcover.)

(f) The localization map $A \to A_f$ induces a map $\operatorname{Spec}(A_f) \to X$. Show that this map gives a *homeomorphism* from $\operatorname{Spec}(A_f)$ to X_f.