Math 493 Fall 2023 HW1

The problems on this homework all involve the dihedral group D_n. This is the group of rigid symmetries of the regular n-gon. It contains $2n$ elements, n rotations and n reflections. Explicitly, if x denotes the rotation by $2\pi/n$ in D_n and y any of the reflections in D_n, then

$$x^n = e, \quad y^2 = e, \quad yxy^{-1} = x^{-1},$$

and

$$D_n = \{e, x, \cdots, x^{n-1}, y, yx, \cdots, yx^{n-1}\}$$

1. Find the lattice of subgroups of D_4. In other words, write out all the subgroups and all the inclusion relations between them. (You should also prove that there aren't any subgroups not on your list.)
2. Likewise, find the lattice of subgroups of D_5.
3. In class, we showed explicitly that D_3 is isomorphic to a subgroup of $\mathrm{GL}_2(\mathbf{R})$. Do the same for D_n.
4. The center of a group G, denoted $Z(G)$, is the set of elements z in G that commute with every element of G. Show that the $Z(G)$ is a subgroup of G. Find $Z(G)$ for $G = D_n$. Describe your answer geometrically.
5. We also showed in class that the dihedral group D_3 is isomorphic to S_3. It is often important to not just know that two groups are isomorphic, but to keep track of the isomorphism. How many ways are there of constructing an isomorphism from D_3 to S_3? Construct all possible isomorphisms and show that there aren't any others.

Note: a homomorphism of groups $\phi : G_1 \to G_2$ is a map satisfying $\phi(ab) = \phi(a)\phi(b)$ for all $a, b \in G_1$. An isomorphism is a bijective homomorphism.