Math 493 Fall 2023 HW2

1. Classify all groups of order six up to isomorphism.
2. Let G be a group. Denote by $\mathrm{Aut}(G)$ the set of automorphisms of G. (An automorphism is an isomorphism from G to itself.) Show that $\mathrm{Aut}(G)$ has the structure of a group (in a natural way). Identify the *group* $\mathrm{Aut}(G)$ for G in the following list:
   (a) C_5
   (b) C_8
   (c) C_{11}
   (d) S_3
   (e) $C_2 \times C_2$
   (f) $\mathbf{Z}$
3. Let p be a prime. Show that up to isomorphism there are exactly two groups of order $2p$, namely the cyclic group C_{2p} and the dihedral group D_p.
4. Let H and K be subgroups of G. Define
$$HK = \{hk : h \in H, k \in K\}.$$
Show that HK is a subgroup of G if and only if $HK = KH$.
5. Let H and K be finite subgroups of a group G. Show that
$$|HK| = \frac{|H| \cdot |K|}{|H \cap K|}$$
Here we are not assuming that HK is a subgroup of G, it may just be a subset; the notation $|HK|$ is still used to denote its cardinality.
6. Let H be a subgroup of G. Define
$$N_G(H) = \{g \in G : gHg^{-1} = H\}. \quad \text{(equivalently: } gH = Hg)$$
Show that $N_G(H)$ is a subgroup of G containing H, that H is normal in $N_G(H)$, and that $N_G(H)$ is the *largest* subgroup of G containing H in which H is normal. (Here *largest* means that it contains any other subgroup with this property.) In particular, H is normal in G if and only if $N_G(H) = G$. **Remark:** $N_G(H)$ is called the normalizer of H in G.
7. **The second isomorphism theorem:** Suppose that H and K are subgroups of G with $H \subseteq N_G(K)$. Show that
   (i) HK is a subgroup of G and K is normal in HK.
   (ii) $H \cap K$ is normal in H.
   (iii) There is a canonical (i.e. natural) isomorphism
$$H/H \cap K \simeq HK/K$$
8. Let $H \trianglelefteq G$ and $K \trianglelefteq G$ be normal subgroups of G with $H \cap K = (e)$.
   (a) Show that any two elements of H and K commute with each other. Namely, $hk = kh$ for any $h \in H$ and $k \in K$.
   (b) Suppose further that $HK = G$. Show that G must be isomorphic to the direct product $H \times K$.