Math 493 Fall 2023 HW3

1. **(15 pts)** Let H and K be subgroups of G.

   (a) (10 pts) Show that the following are equivalent:

      i. H and K are normal in G and every element in G can be written uniquely as a product hk, with $h \in H$ and $k \in K$.

      ii. H and K are normal in G, $G = HK$ and $H \cap K = (e)$.

      iii. The natural map $H \times K \to G$, $(h, k) \mapsto hk$ is an isomorphism of groups.

   In this case we say that G is the *internal* direct product of the (normal) subgroups H and K.

   (b) (5 pts) What is the relation between internal and external direct products? Discuss ...

2. **(15 pts)** Suppose that N is a normal subgroup of G, and let $\phi : G \to G/N$ denote the quotient map.

   (a) (10 pts) Show that there is a bijection

$$\{\text{subgroups of } G \text{ containing } N\} \leftrightarrow \{\text{subgroups of } G/N\}$$

   given by

$$H \mapsto \phi(H) = H/N$$

   for H a subgroup of G, with the map in the other direction being given by

$$\bar{H} \mapsto \phi^{-1}(\bar{H})$$

   for $\bar{H}$ a subgroup of G/N. In this bijection, show that normal subgroups of G correspond to normal subgroups of G/N.

   (b) (5 pts) **The third isomorphism theorem:** Suppose that H is a normal subgroup of G containing N. Show that N is normal in H, that H/N is a normal subgroup of G/N and that there is a natural isomorphism

$$\frac{G/N}{H/N} \simeq G/H.$$

3. **(10 pts)** Classify all abelian groups of order 8. Show that any such group must be isomorphic to one of C_8, $C_4 \times C_2$, $C_2 \times C_2 \times C_2$.

4. **(20 pts)** In this problem we will classify the nonabelian groups of order 8.

   (a) (5 pts) Let G be a nonabelian group of order 8. Argue that G must contain an element x of order 4. Let $y \in G \setminus \langle x \rangle$. Then

$$G = \{e, x, x^2, x^3, y, yx, yx^2, yx^3\}.$$

   The element yxy^{-1} must be equal to one of the elements in the list above. Which one? Likewise, what can you say about $y^{-1}xy$?

   (b) (10 pts) Now consider the order of y. Since G is nonabelian, we must have that $o(y)$ is either 2 or 4. If $o(y) = 2$, then $G \simeq D_4$. If $o(y) = 4$, show that $y^2 = x^2$ and that G must be isomorphic to the subgroup Q of $\mathrm{GL}_2(\mathbf{C})$ generated by the matrices

$$a := \begin{bmatrix} i & \\ & -i \end{bmatrix}, \quad b := \begin{bmatrix} & -1 \\ 1 & \end{bmatrix}.$$

The group Q is called the *quaternion group*.

Actually the elements a and b play completely symmetric roles, so it is nicer (and more traditional) to rename the elements in this group as follows:

$$\mathbf{i} := \begin{bmatrix} i & \\ & -i \end{bmatrix}, \quad \mathbf{j} = \begin{bmatrix} & -1 \\ 1 & \end{bmatrix}, \quad \mathbf{k} := \mathbf{ij}.$$

Then

$$G = \{\pm\mathbf{1}, \pm\mathbf{i}, \pm\mathbf{j}, \pm\mathbf{k}\}$$

and we have

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -\mathbf{1}, \quad \mathbf{ij} = \mathbf{k} = -\mathbf{ji}, \quad \mathbf{jk} = \mathbf{i} = -\mathbf{kj}, \quad \mathbf{ki} = \mathbf{j} = -\mathbf{ik}.$$

(c) (5 pts) Using the $\mathbf{i}, \mathbf{j}, \mathbf{k}$ notation, write out the lattice of subgroups of Q. Which of these subgroups are normal? What is the center of Q?

5. **(10 pts)** (Artin, Ex. 2.12.2) In the group $G = \mathrm{GL}_3(\mathbf{R})$, consider the subsets

$$H = \begin{bmatrix} 1 & * & * \\ & 1 & * \\ & & 1 \end{bmatrix}, \quad K = \begin{bmatrix} 1 & & * \\ & 1 & \\ & & 1 \end{bmatrix}$$

where $*$ represents any real number and the empty spaces represent zeros. Show that H is a subgroup of G, that K is a normal subgroup of H, and identify the quotient group H/K. Also, determine the center of H.

6. **(10 pts)** At this point, we have covered most of Chapter 2 of Artin. Read the chapter, and write a short summary of any one topic from the chapter that we did not cover in detail in class.

7. **(10 pts)** Describe all the conjugacy classes in the dihedral group D_n. Write down the class equation.

8. **(10 pts)** Describe all the conjugacy classes in the quaternion group Q. Write down the class equation.