Math 493 Fall 2023 HW4

You may use the Sylow theorems on this homework set.

1. **(10pts)** Let p be a prime. Classify the groups of order p^2 up to isomorphism.

2. **(Internal) Semi-direct products.** Let G be a group, H and K subgroups of G with H normal in G.

   (a) **(10 pts)** Show that the following are equivalent:

      i. Every element g in G can be written uniquely as hk with $h \in H$ and $k \in K$.

      ii. $G = HK$ and $H \cap K = (e)$.

      iii. The composite map $K \to G \to G/H$ is an isomorphism.

   If any (and therefore all) of these three equivalent properties hold, we say that G is the (internal) semi-direct product of H and K; this is symbolically written as $G = H \rtimes K$. (The symbol is supposed to indicate that H is normal in G, since the roles of H and K are not symmetric any more, as opposed to the case of direct products.)

   (b) **(5 pts)** Suppose that $G = H \rtimes K$. Show that G may be identified as a *set* with the Cartesian product $H \times K$, and that viewed this way, the multiplication on G is given by

   $$(h_1, k_1) \cdot (h_2, k_2) = \big(h_1(k_1 h_2 k_1^{-1}), k_1 k_2\big) = (h_1 \phi_{k_1}(h_2), k_1 k_2),$$

   where for any $k \in K$, $\phi_k : H \to H$ is the map given by $\phi_k(h) = khk^{-1}$.

   (c) **(5 pts)** Show that $\phi_k \in \mathrm{Aut}(H)$ and that the map $k \mapsto \phi_k$ gives a *homomorphism*

   $$\phi : K \to \mathrm{Aut}(H).$$

   How would you use ϕ to characterize whether G is the (internal) direct product of H and K?

3. **(External) semi-direct products.** The previous problem motivates the following definition. Let H and K be abstract groups, and let

   $$\phi : K \to \mathrm{Aut}(H), \quad k \mapsto \phi_k$$

   be a homomorphism. The (external) semi-direct product $H \rtimes_\phi K$ is defined as follows. As a set, it is the Cartesian product $H \times K$, with group law given by

   $$(h_1, k_1) \cdot (h_2, k_2) = (h_1 \phi_{k_1}(h_2), k_1 k_2).$$

   (a) **(5 pts)** Verify that this gives a group. Namely, identify the identity element, show that the group law is associative, and that inverses exist.

   (b) **(5 pts)** What is the relation between internal and external semi-direct products? Discuss ...

   (c) **(10 pts)** It is rather tricky to determine when two different choices of ϕ (for the same H and K) give rise to isomorphic groups. However, at least we can give a *sufficient* criterion for two semi-direct products to be isomorphic, as follows. The set of ϕ above, namely the set $\mathrm{Hom}(K, \mathrm{Aut}(H))$, carries an action of $\mathrm{Aut}(H) \times \mathrm{Aut}(K)$: if $(\psi, \mu) \in \mathrm{Aut}(H) \times \mathrm{Aut}(K)$, then

   $$((\psi, \mu) \cdot \phi)(k) = \psi \circ (\phi(\mu^{-1}(k)) \circ \psi^{-1}.$$

   Verify that this is an action. Show that if ϕ and ϕ' are in the same orbit for this action, then

   $$H \rtimes_\phi K \simeq H \rtimes_{\phi'} K.$$

4. **(10 pts)** Show that up to isomorphism there are exactly two groups of order 21.
5. **(10 pts)** Classify all the groups G of order 30 up to isomorphism. (Hint: First show that G must have a normal subgroup of order 15.)
6. **(20 pts)** Use what you know about semi-direct products to classify the groups G of order 12 up to isomorphism. (Hint: First show that G must have either a normal 2-Sylow subgroup or a normal 3-Sylow subgroup, hence is a semi-direct product $H \rtimes K$, where either H is of order 3 or of order 4. Then work through all the possibilities for H, K, $\phi : K \to \mathrm{Aut}(H)$, deciding which of these give isomorphic groups.)
7. **(10 pts)** Let H be a p-Sylow subgroup of a finite group G. Show that

$$N_G(N_G(H)) = N_G(H).$$