Math 493 Fall 2023 HW6

1. (**10 pts**) Find *all* composition series and the corresponding simple factors for S_4.

2. (**20 pts**) Find the derived series (i.e. , the commutator series) for the following groups and identify which of them are solvable:

   (a) (10 pts) All the groups of order 12. (You should use the classification you worked out in a previous homework.)

   (b) (10 pts) S_n for all n.

3. (**5 pts**) Let G be the group
$$G = \mathrm{PSL}_2(\mathbf{F}_5) = \mathrm{SL}_2(\mathbf{F}_5)/\{\pm I\}.$$
Show that $G \simeq A_5$.

4. **(40 pts) A simple group of order** 168**.** Let G be the group
$$G = \mathrm{PSL}_2(\mathbf{F}_7) = \mathrm{SL}_2(\mathbf{F}_7)/\{\pm I\}.$$

   (a) (5 pts) Show that $|G| = 168 = 2^3 \cdot 3 \cdot 7$.

   (b) (5 pts) Let x and y be the classes of the matrices
$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$
in G. Show that x and y generate distinct 7-Sylow subgroups H and K respectively. Deduce that $n_7(G) = 8$.

   (c) (5 pts) Let $N \trianglelefteq G$ be a normal subgroup. Show that $[G:N] \neq 2$. (Hint: Suppose that $[G:N] = 2$. How many 7-Sylows would such an N have?)

   (d) (5 pts) Let $N \trianglelefteq G$ be a normal subgroup. Show that $[G:N] \neq 3$. (Hint: Suppose that $[G:N] = 3$. Show that N must contain both H and K. In particular it contains the element $u = xy^2x$. Then show that u has order 3 in G. Why is this a contradiction?)

   (e) (5 pts) Let $N \trianglelefteq G$ be a normal subgroup. Show that $[G:N] \neq 6$.

   Suppose now that G is not simple. We will derive a contradiction.

   (f) (5 pts) Let $N \trianglelefteq G$ be a *maximal* proper normal subgroup. Show that the only possibility for its index in G is $[G:N] = 7$, so that $|N| = 24$.

   (g) (5 pts) Since $|N| = 24$, we must have $n_2(N) = 1$ or 3. Show that $n_2(N) \neq 1$. (Hint: If $n_2(N) = 1$, show that the unique 2-Sylow subgroup H of N is normal in G. Then G/H is a group of order 21, and has a normal 7-Sylow subgroup $\bar{K}$. Now $\bar{K}$ corresponds to a normal subgroup K of G containing H. What is $[G:K]$?)

   (h) (5 pts) Show that $n_2(N) \neq 3$ either, and thus G must be simple. (Hint: Suppose $n_2(N) = 3$. Let H_1, H_2, H_3 denote the 2-Sylows in N, which are the same as the 2-Sylows in G. Consider the conjugation action of G on $\{H_1, H_2, H_3\}$. This gives a nontrivial homomorphism $\varphi : G \to S_3$. Now what is the index of $\ker(\varphi)$ in G?)

5. **(15 pts) A(nother?) simple group of order** 168**.** Let $\mathscr{G}$ be the group

$$\mathscr{G} = \mathrm{GL}_3(\mathbf{F}_2).$$

   (a) (5 pts) Show that $|\mathscr{G}| = 168$.

   (b) (5 pts) Show that $\mathscr{G}$ is simple. (Hint: Explain why the same argument as in the previous problem works, taking

$$x = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad y = x^T = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

   and $u = yx = x^T x$.)

   (c) (5 pts) Identify the isomorphism class of a 2-Sylow subgroup in $\mathscr{G}$.

6. (**10 pts**) One can show that up to isomorphism, there is a unique simple group of order 168, so that in fact the groups G and $\mathscr{G}$ from the previous two problems are isomorphic. This group has some beautiful connections with number theory and algebraic geometry. In particular it is the automorphism group of the *Klein quartic*:

$$X^3Y + Y^3Z + Z^3X = 0.$$

The equation above defines an *algebraic curve* of genus 3 in the projective space $\mathbf{P}^2$. Over $\mathbf{C}$, it may be thought of as a Riemann surface, and topologically it is a sphere with three handles attached. Now there is a theorem in algebraic geometry that if C is a curve of genus $g \geq 2$ over the complex numbers, then its (algebraic) automorphism group $\mathrm{Aut}(C)$ is a finite group of order $\leq 84(g-1)$. If $g = 3$, this gives an upper bound of 168, and this upper bound is exactly realized for the curve C above, its automorphism group being isomorphic to G (or $\mathscr{G}$). While it is somewhat tricky to see that $\mathrm{Aut}(C) \simeq G$, we can amuse ourselves with the following exercises:

   (a) (3 pts) Construct a nontrivial element σ in $\mathrm{Aut}(C)$ of order 3. (Hint: you want to define σ by a formula of the sort $(X, Y, Z) \mapsto (F_1(X, Y, Z), F_2(X, Y, Z), F_3(X, Y, Z))$ where F_1, F_2, F_3 are polynomials in the variables X, Y, Z.)

   (b) (3 pts) Construct a nontrivial element τ in $\mathrm{Aut}(C)$ of order 7. (Hint: try to use a 7th root of unity.)

   (c) (4 pts) Consider the subgroup of $\mathrm{Aut}(C)$ generated by σ and τ. What can you say about the order of this group and its structure? (Hint: semi-direct products?)

Historical Note: The classification of finite simple groups is one of the great achievements of 20th century mathematics, and the original proof took over ten thousand pages of journal articles. A key idea in this work was a proposal by Richard Brauer[1] to study simple groups by using centralizers of an involution.

An involution is an element x in G of order 2, and Brauer's idea was to characterize for a given group H, the possible simple groups G such that $Z_G(x) \simeq H$ for some involution $x \in G$. Of course for this idea to work, one would need to know that any nonabelian simple group has an involution. This was provided by the following landmark result of Feit and Thompson:

Theorem 0.1. *(Feit-Thompson, 1962) If G is a nonabelian simple group, then $|G|$ is even.*

The proof of this theorem (which takes more than 150 pages) gave hope that a full classification would eventually be possible.

The classification (which was thought to be completed in the early 1980's but was eventually finished some years later) runs as follows: every finite simple group is of one of four types:

   (a) The cyclic group C_p for p prime.

   (b) The alternating group A_n for $n \geq 6$.

[1] Brauer was a professor at Toronto until 1948, then at UM from 1948-1952 before moving to Harvard. One of his students from Toronto was Nesbitt, after whom the Nesbitt undergraduate lounge in our department is named. Nesbitt had moved earlier to UM and ran the actuarial program here from 1938 to 1980.

(c) A finite simple group of Lie type: these break up into further subfamilies.

eg. one of these subfamilies is $\mathrm{PSL}_n(F) = \mathrm{SL}_n(F)/Z(\mathrm{SL}_n(F))$ where F is a finite field. These are simple for all finite fields F when $n \geq 3$ and for all finite fields F with at least 4 elements if $n = 2$. Note that the examples we saw in the problems above are all of this type: $\mathrm{PSL}_2(\mathbf{F}_5)$, $\mathrm{PSL}_2(\mathbf{F}_7)$ and $\mathrm{GL}_3(\mathbf{F}_2) \simeq \mathrm{PSL}_3(\mathbf{F}_2)$. Occasionally, there are "exceptional isomorphisms" between members of different families. For example, $A_5 \simeq \mathrm{PSL}_2(\mathbf{F}_5)$, $A_6 \simeq \mathrm{PSL}_2(\mathbf{F}_9)$, $A_8 \simeq \mathrm{PSL}_4(\mathbf{F}_2)$.

(d) One of 26 "sporadic groups". The largest of these is called the monster M and has order about 8×10^{53}.

Another landmark result (which marked in some sense the beginning of the end of the classification project) was the discovery and construction of the monster group M. The existence of the monster was predicted by Bernd Fischer and Robert Griess[2] in 1973. A few months later, Griess computed the predicted order of the monster. Nevertheless, there was at that time no proof that such a group actually existed. The monster was first constructed by Griess in a remarkable piece of work in late 1979 - early 1980.

You can find the journal article with the construction here: The Friendly Giant

Here is the MathSciNet review of it, which gives some idea of the method and tools used.

You can read about the some of the history of the sporadic groups and the work leading up to the construction in Griess's own words in this survey article. The most relevant sections are 14-18. Also, on p. 37 of the article you can find a table of the orders of all the nonabelian finite simple groups, including all the families of Lie type.

The monster has remarkable connections to other areas of mathematics, especially number theory. One of these connections is a conjecture called Monstrous Moonshine, which was conjectured in the late 70's and proved by Richard Borcherds in the early 90's.

If any of the above piqued your interest, here are some classes we offer where you can learn more:

1. To learn more about Algebraic curves, you can try Math 631 (Algebraic Goemetry)
2. To learn about Riemann surfaces, you will want to start with Math 596 (Complex Analysis)
3. To learn about groups of Lie type, you can take Math 538 (Lie Algebras) and Math 637 (Lie Groups)

[2] Prof. Griess has been a member of the UM faculty since 1971.