Math 493 Fall 2023 HW7

0. (**10 bonus points**) Work on at least part of this homework with two other students from this class that you have not worked with previously. Describe which problems you worked on together. Also, include one fact about each of these students that you did not know about them.

1. (**50 pts, Artin Ex. 7.11.3.**) Use the Todd-Coxeter Algorithm to analyze the group generated by two elements x, y, with the following relations. Determine the order of the group and identify the group if you can:

   (a) $x^2 = y^2 = e, xyx = yxy$
   (b) $x^3 = y^3 = e, xyx = yxy$
   (c) $x^4 = y^2 = e, xyx = yxy$
   (d) $x^4 = y^4 = x^2y^2 = e$
   (e) $x^3 = e, y^2 = e, yxyxy = e$
   (f) $x^3 = y^3 = yxyxy = e$
   (g) $x^4 = e, y^3 = e, xy = y^2x$
   (h) $x^7 = e, y^3 = e, yx = x^2y$
   (i) $x^{-1}yx = y^{-1}, y^{-1}xy = x^{-1}$
   (j) $y^3 = e, x^2yxy = e$

2. (**10pts**) Analyze the dihedral group: the group generated by x, y, subject to

$$x^n = e, y^2 = e, xyxy = e,$$

   using the Todd-Coxeter algorithm. Show directly that this is a group of order $2n$. (Previously, we used a matrix representation to show that this group has order $2n$, so the point of the exercise is to do this directly, without using the matrix representation.)

3. (**10 pts, Artin Ex. 11.5**) Let G be the group generated by elements x, y, with relations $x^4 = e, y^3 = e, x^2 = yxy$. Prove that this group is trivial in two ways: using the Todd-Coxeter Algorithm, and working directly with the relations.

4. (**20 pts, Artin Ex. 7.11.5.**)

   (a) Prove that the group G generated by elements x, y, z with relations $x^2 = y^3 = z^5 = e, xyz = e$ has order 60.
   (b) Let H be the subgroup generated by x and zyz^{-1}. Determine the permutation representation of G on G/H, and identify H.
   (c) Prove that G is isomorphic to the alternating group A_5.
   (d) Let K be the subgroup of G generated by x and yxz. Determine the permutation representation of G on G/K, and identify K.

5. (**10 points, Artin Ex. 7.10.10**) Let F be the free group on $\{x, y\}$ and let R be the smallest normal subgroup containing the commutator $xyx^{-1}y^{-1}$.

   (a) Show that $x^2y^2x^{-2}y^{-2}$ is in R.
   (b) Prove that R is the commutator subgroup of F.