Math 493 Fall 2023 HW9

Please do not submit this assignment. These problems are for reviewing for the final exam.
G is a finite group in all the problems below.

1. Let $\mathrm{Fun}(G, \mathbf{C})$ denote the vector space of $\mathbf{C}$-valued functions on G.
   (a) For (ρ, V) a representation of G, let $P = (p_{ij}(g))$ denote the matrix of $\rho(g)$ with respect to some basis $\mathfrak{B}$ of V. Show that the subspace of $\mathrm{Fun}(G, \mathbf{C})$ spanned by the functions $p_{ij}(g)$ is independent of the choice of basis $\mathfrak{B}$ and only depends on the isomorphism class of ρ .
   (b) Show that these matrix coefficients $p_{ij}(g)$, as (ρ, V) vary over all the distinct irreducible representations of G, form a basis for $\mathrm{Fun}(G, \mathbf{C})$.
2. Let (ρ_i, V_i) be the distinct irreducibles of G. We have seen that given any finite dimensional representation (ρ, V), it is the direct sum
$$V = \oplus W_j \tag{1}$$
of irreducibles, each of which must be isomorphic to one of the ρ_i. We have also seen that in such a decomposition, the number of W_j that are isomorphic to a fixed (ρ_i, V_i) is independent of the choice of decomposition. However the actual decomposition (i.e. the physical subspaces W_j) may not be unique. Nevertheless, show that the decomposition above does satisfy the following uniqueness property:
   Fix one of the ρ_i and let W be the (direct) sum of those W_j that are all isomorphic to ρ_i. Then the subspace W of V thus obtained depends only on the isomorphism class of ρ_i and is independent of the choice of decomposition (1). (This subspace W is called the ρ_i-isotypic part of V.)
   Hint: Let χ_i be the character of ρ_i, which is a class function on G. Let $p_i : V \to V$ be the map given by
$$p_i(v) = n_i \rho_{\chi_i}$$
   in the notation of Prop. 0.6 of the "group rep 5" note., where $n_i = \dim(V_i)$. Show that p_i must be the identity on all the W_j that are isomorphic to our fixed ρ_i and is zero on all the W_j that are not isomorphic to the fixed ρ_i. Thus p_i is a projection to W. But p_i only depends on ρ and the isomorphism class of ρ_i.
3. Construct the character tables for the three nonabelian groups of order 12.
4. **The group** S_4
   (a) Construct the character table of S_4, using just what you know about the character table of S_3 and the properties of characters we have proved. *Hint:* Write S_4 as a semidirect product of a normal subgroup of order 4 and S_3.
   (b) Verify that the rows of the character table satisfy the expected orthogonality relations.
   (c) Consider the natural permutation action of S_4 on $\mathbf{C}^4$ (i.e. $\sigma(e_i) = e_{\sigma(i)}$.) Describe how this representation decomposes as a sum of irreducibles.
   (d) For each irreducible representation of S_4 (corresponding to a character in the character table of the previous problem), describe how its restriction to A_4 decomposes as a sum of representations of A_4.
5. Explain how you can use the character table of G to determine all the normal subgroups of G.
6. Artin Ex. 10.4.9
7. Artin Ex. 10.4.10
8. Artin Ex. 10.4.11