

Math 614: Problem Set 1

Due Thursday September 19, 2019

1. Let \mathcal{C} be a category. Associate a directed graph $\Gamma_{\mathcal{C}}$ to \mathcal{C} as follows (freely assume the axiom of choice). The vertices of $\Gamma_{\mathcal{C}}$ are the isomorphism types of objects in \mathcal{C} , with exactly one arrow between distinct vertices for every morphism between the corresponding objects, up to the following equivalence: given morphisms $f : A \rightarrow B$ and $g : A' \rightarrow B'$, we say that f and g are equivalent if there exist isomorphisms α and β such that the following diagram commutes in \mathcal{C} :

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow \alpha & & \downarrow \beta \\ A' & \xrightarrow{g} & B' \end{array}$$

Describe the directed graphs of the following categories:

1. The category of finite dimensional real vector spaces.
 2. The full subcategory of the category of abelian groups consisting of the simple abelian groups.
 3. The full subcategory of the category of comm rings with 1 consisting of all quotient rings of \mathbb{Z} .
2. Let $R = \mathbb{Z}[X]$ be a polynomial ring over \mathbb{Z} . Let A be the subring generated by $x^2 - 3$ and $2x$ and let I be the ideal generated by $x^2 - 3$ and $2x$. Describe the ring R/I as an abelian group; in particular, find a minimal set of generators and a decomposition into cyclic \mathbb{Z} -modules. What is its cardinality? Do the same for the abelian group R/A .
3. Let X be a topological space, and let $x \in X$ be a fixed point. Let S be the set of real-valued continuous functions defined on a open neighborhood of x , and define an equivalence relation on S as follows: $f \equiv g$ if there is some smaller open neighborhood of x on which the restrictions of f and g agree. Let T be the set of all equivalence classes of elements in S . Prove that T has a natural structure of a **local ring**, ie, a ring with a unique maximal ideal m , and determine the residue class field T/m . In the case where $X = \mathbb{R}$ with its usual Euclidean topology, determine whether or not T is Noetherian.
4. Let $p_1, \dots, p_n \dots$ be an infinite strictly increasing sequence of positive prime integers. Let $K_n = \mathbb{Z}/p_n$, and set $K_0 = \mathbb{Q}$, the rational numbers. Let R denote the subring of the product ring $\prod_{i=0}^{\infty} K_i$ consisting of sequences $a_0, a_1, \dots, a_n \dots$ such that for all sufficiently large n , a_n is the image of a_0 in K_n (this makes sense because if $a_0 = r/s$, where $r, s \in \mathbb{Z}, s \neq 0$, then s is not divisible by p_n for n sufficiently large). Describe $\text{Spec } R$ and its Zariski topology. Is R Noetherian?
- 5.* Let P and Q be prime ideals of a ring R . Show that there is no prime ideal contained in both P and Q if and only if P and Q have disjoint open neighborhoods in $\text{Spec } R$. Is the subspace $\text{minSpec } R$ of $\text{Spec } R$ consisting of all *minimal* primes of R Hausdorff?
- 6.* Let k be a field. Consider the subalgebra R of the polynomial ring $k[x, y]$ generated by all monomials of the form $x^a y^b$ where $a/b \leq \lambda$ for some fixed positive $\lambda \in \mathbb{R}$. Prove that R is finitely generated (and hence Noetherian) over k if and only if $\lambda \in \mathbb{Q}$.