Math 614: Problem Set 2

Due Tuesday October 8, 2019

1. Let $\mathscr{C}$ be a category. For a fixed object $X \in Ob(\mathscr{C})$, define a functor $h_X : \mathscr{C} \to Set$ sending $Y \mapsto \mathrm{Mor}_{\mathscr{C}}(X, Y)$.

1. For the category $\mathbb{Z}$-mod, and the object $X = \mathbb{Z}/\langle 12\rangle$, describe $h_X(\mathbb{Z}/\langle 24\rangle \oplus \mathbb{Z}/\langle 7\rangle \oplus \mathbb{Q})$ explicitly. In particular, what is its cardinality?
2. Given a morphism $Y \xrightarrow{g} X$, define, for each $Z \in \mathscr{C}$, a map of sets ${}^gT_Z$ by
$$ {}^gT_Z(X) : \mathrm{Mor}_{\mathscr{C}}(X, Z) \longrightarrow \mathrm{Mor}_{\mathscr{C}}(Y, Z) \quad f \mapsto f \circ g. $$
Verify that ${}^gT_Z$ defines a natural transformation from h_X to h_Y.
3. Show that there is a bijection between $\mathrm{Mor}(Y, X)$ and the set of all natural transformations from h_X to h_Y.
4. A natural transformation T between two functors $F, G : \mathscr{D} \to \mathscr{D}'$ is an **isomorphism** of functors, if, for every object $X \in \mathscr{D}$, the morphism $T(X) : F(X) \to G(X)$ is an isomorphism in $\mathscr{D}'$. Show that h_X and h_Y are isomorphic functors if and only if X and Y are isomorphic objects.
5. Show that if a covariant functor is representable, then the object representing it is unique, up to isomorphism.

2. For each prime ideal P of a ring R, define $P^{(n)}$ as $P^n R_P \cap R$. Let $T = K[x_1, \dots, x_n, y_1, \dots, y_n]$ be the polynomial ring in $2n$ variables, and let $S = T/\langle \sum_{i=1}^n x_i y_i \rangle$. Compute the following

(a) For $P = \langle x_1, \dots, x_n, y_1, \dots, y_n\rangle \subset T$, compute $P^{(4)}$.

(b) For $P = \langle x_1, \dots, x_n, y_1, \dots, y_{n-1}\rangle \subset T$, compute $P^{(3)}$.

(c) For $P = \langle x_1, \dots, x_n, y_1, \dots, y_{n-1}\rangle \subset S$, compute $P^{(2)}$. Here we abuse notation, using elements of T to represent their classes in S.

3. Let R be a subring of S. Suppose that there is an R-module map $\theta : S \to R$ such that $\theta(1) = 1$.

(a) Show that for every ideal of R, $IS \cap R = I$.

(b) Show that the map Spec $S \to$ Spec R is surjective.

(c) Let G be a finite group, and suppose G acts on a commutative ring S by ring automorphisms. Let $S^G = \{f \in S \mid g \cdot f = f\}$. Assuming that $|G|$ is a unit in S, prove or disprove that induced map of Spectra is surjective.

(d) (*) Same as (c) without the assumption that $|G|$ is a unit in S.

4. Let $\zeta \in \mathbb{C}$ be a primitive n-th root of unity. Define an action of the cyclic group $C_n = \{\zeta^i \mid i \in \mathbb{N}\} \subset \mathbb{C}^\times$ by $\mathbb{C}$-algebra automorphisms on the polynomial ring $\mathbb{C}[x, y]$ by declaring $\zeta \cdot f(x, y) = f(\zeta^{-1}x, \zeta^{-1}y)$.

(a) Describe the induced action of C_n on maxSpec $\mathbb{C}[x,y]$ by explicitly explaining the action on an arbitrary maximal ideal. Describe the corresponding action on $\mathbb{C}^2$ induced by the Nullstellensatz.

(b) Describe the ring of invariants $\mathbb{C}[x,y]^{C_n} = \{f \in \mathbb{C}[x,y] \mid g \cdot f = f \ \forall g \in C_n\}$ as a subring of $\mathbb{C}[x,y]$. [HINT: It may be helpful to think about the degrees of polynomials.]

(c) Show that $\mathbb{C}[x,y]^{C_n}$ is finitely generated over $\mathbb{C}$ by $n+1$ elements.

(d) Find a presentation for $\mathbb{C}[x,y]^{C_n}$ in the case $n=2$.

(e) * For the ring inclusion $\mathbb{C}[x,y]^{C_n} \subset \mathbb{C}[x,y]$, show that the induced map maxSpec $\mathbb{C}[x,y] \to$ maxSpec $\mathbb{C}[x,y]^{C_n}$ can be identified with the quotient map $\mathbb{C}^2 \to V$, where V is the set of orbits of the action of C_n on $\mathbb{C}^2$ described in (a).

5.* For any vector space V over K, let $\mathbb{P}(V)$ denote the set of all one-dimensional subspaces of V. When V is the space of column vectors K^{n+1}, a point P in $\mathbb{P}(V)$ can be represented by a (column) vector v spanning P; this is well-defined only up to non-zero scalar multiple. Define a map

$$\Sigma : \mathbb{P}(K^{n+1}) \times \mathbb{P}(K^{m+1}) \to \mathbb{P}(K^{(n+1)\times(m+1)})$$

sending $(v,w) \mapsto v w^{tr}$, the (one-dimensional space spanned by the) matrix product of the column vector v with the row vector w^{tr}.

1. Prove that Σ is well-defined and injective.
2. Find polynomials in $(m+1)(n+1)$ variables which vanish on a point P in $\mathbb{P}(K^{(n+1)\times(m+1)})$ if and only if P is in the image of Σ.
3. Let $\pi : \mathrm{im}(\Sigma) \to \mathbb{P}(K^{n+1})$ (respectively, $\psi : \mathrm{im}(\Sigma) \to \mathbb{P}(K^{m+1})$ be defined by taking a representative $Q \in K^{(n+1)\times(m+1)}$ to any of its columns (respectively rows). Show that these maps are well-defined and that $\mathrm{im}(\Sigma) \xrightarrow{\pi,\psi} \mathbb{P}(K^{n+1}) \times \mathbb{P}(K^{m+1})$ defines an inverse map to Σ.

6.* The ring of complex analytic germs in d variables, denoted $\mathbb{C}\{z_1,\ldots,z_d\}$, is the subring of $\mathbb{C}[[z_1,\ldots,z_d]]$ consisting of power series that converge on some ball containing the origin.

- A Weierstrass polynomial of degree t in z_d is a function of the form $z^t + f_{t-1}z^{t-1} + \cdots + f_0$ with $f_0,\ldots,f_{t-1} \in \mathbb{C}\{z_1,\ldots,z_{d-1}\}$.
- The Weierstrass preparation theorem says that: If $f \in \mathbb{C}\{z_1,\ldots,z_d\}$, satisfies $f(0,...,0)=0$, and $f(0,...,0,z_d) \neq 0$, then there is some unit $g \in \mathbb{C}\{z_1,\ldots,z_d\}$, and Weierstrass polynomial h in z_d such that $f = gh$.
- The Weierstrass division theorem says that if h is a Weierstrass polynomial of degree t in z_d, and $f \in \mathbb{C}\{z_1,\ldots,z_d\}$, then $f = ph + q$ for some $p \in \mathbb{C}\{z_1,\ldots,z_d\}$, and $q \in \mathbb{C}\{z_1,\ldots,z_{d-1}\}[z_d]$ of degree less than t in z_d.

Use the Weierstrass preparation theorem and Weierstrass division theorem to show that $\mathbb{C}\{z_1,\ldots,z_d\}$ is Noetherian.