Math 614: Problem Set 3

Due Thursday October 31, 2019

1. Let $\mathcal{C}$ be a category. Let $\{M_\lambda\}$ be a collection of objects in $\mathcal{C}$, indexed by the set Λ. An object M of $\mathcal{C}$ is said to be the **product of** $\{M_\lambda\}$ **in** $\mathcal{C}$ if for all $\lambda \in \Lambda$ there are morphisms $\pi_\lambda : M \to M_\lambda$ in $\mathcal{C}$ satisfying the following universal property: For any object $N \in \mathcal{C}$ with morphisms $\mu_\lambda : N \to M_\lambda$ for all $\lambda \in \Lambda$, there exists a unique morphism $f : N \to M$ such that $\pi_\lambda \circ f = \mu_\lambda$ for all $\lambda \in \Lambda$. The **coproduct** is defined similarly, but with all arrows reversed.

For each of the categories below, describe the product of an arbitrary collection of objects or explain why it does not exist. For at least one of these categories, describe also the coproduct, or explain why it does not exist.

1. The category of commutative rings with identity.
2. Fix a set X, and let $\mathcal{C}$ be the category of its subsets; define $Mor(E, F)$ to be empty unless $E \subset F$ and the one map $E \subset F$ otherwise.
3. Fix a topological space X, and let $\mathcal{O}p$ be the category of its open subsets, where again, the only morphisms are the inclusions $U \subset V$.

2. Consider the ring extension $\iota : \mathbb{C}[x, y] \hookrightarrow \mathbb{C}[x, y, z]/\langle z^3 - 3xz^2 + y\rangle$. Let $\pi : Y \to X$ be the induced map on Spec.

1. Prove that every fiber of π is non-empty and of cardinality at most three.
2. Prove that the fiber of π over a closed point P consists of three distinct points if and only if P is in the open set $D(f)$ where f is some explicit polynomial in $\mathbb{C}[x, y]$ (find it!).
3. Find an explicit map $\mathbb{C}^3 \to \mathbb{C}^2$ whose restriction to the algebraic set $\mathbb{V}(z^3-3xz^2+y)$ in $\mathbb{C}^3$ corresponds precisely to the restriction of π to the subspace of maximal ideals in Y under the correspondence given by Hillbert's Nullstellensatz.

3. Let K be an algebraically closed field.

1. Let $f : K^2 \to K^2$ be defined by $f(x, y) = (x, 1 + xy)$ for all $x, y \in K$. Find the image of f, and show that it is neither open nor closed in K^2 (in the Zariski topology).
2. Let $g : K^2 \to K^2$ be defined by $g(x, y) = (x + y(1 + xy), 1 + xy)$ for all $x, y \in K$. Find the image U of g, and show that it is open in K^2. Describe the sets $A_i \subset U$ of points P such that $g^{-1}(P)$ has exactly i elements for $i = 1, 2$.

4. Let $T = K[x_1, \ldots, x_n]$ be a polynomial ring over a field K, and let f denote the sum of the square-free products of the variables taken $n - 1$ at a time. Let $R = T[1/f]$. Explicitly express R as a module-finite extension of a polynomial ring over K. In particular, give the algebraically independent generators of the polynomial ring explicitly.

5.* Let R be a ring such that R_m is Noetherian for all $m \in \text{maxSpec } R$ and for each non-zero $x \in R$, the set $\mathbb{V}(x)$ is finite. Prove that R is Noetherian.