Math 614: Problem Set 4

Due Tuesday November 19, 2019

1. Let S be an R-algebra, let M be an R-module and let N be an S-module.

1. Show that there is a natural S-module isomorphism $\operatorname{Hom}_R(M, N) \cong \operatorname{Hom}_S(S \otimes_R M, N)$.
2. Show that $S \otimes_R M$ represents the functor $\operatorname{Hom}_R(M, -)$ from S-mod to S-mod.

2. Let R be an $\mathbb{F}_p$-algebra, where $p > 0$ is prime.

1. Show that the map $F : R \to R$ sending $r \mapsto r^p$ is a ring homomorphism.
2. Show that F is module finite if and only if F is algebra finite.
3. Show that the induced map on Spec for the map F in (a) is the identity map.
4. In the case that $R = \mathbb{F}_p[x_1, \ldots, x_d]$, prove that R is free as an R-module when viewed via restriction of scalars from F, and find its rank.

3. Let R be a K-algebra, where K is a field. Let $J, I_1, \ldots, I_t$ be ideals of R. Suppose that $J \subset I_1 \bigcup I_2 \bigcup \cdots \bigcup I_t$.

1. Assuming K is infinite, show that $J \subset I_j$ for some $j = 1, 2, \ldots, t-1$ or t.
2. Without assuming K is infinite, but assuming $I_1, \ldots, I_{t-2}$ are prime, show that $J \subset I_j$ for some $j = 1, 2, \ldots, t-1$ or t.

4. Let R be a ring and let $P_1, P_2 \ldots, P_t$ be prime ideals.

1. Let $U = R \setminus (\bigcup_{i=1}^{t} P_i)$. Show that U is a multiplicative system.
2. Show that $U^{-1}R$ has finitely many maximal ideals and describe them explicitly.
3. Fix a natural number $t > 1$ and $d_1 < d_2 < \ldots < d_t$. Construct Noetherian domain S which admits exactly t maximal ideals of heights $d_1, d_2, \ldots, d_t$, respectively.

5. Consider a system of m linear equations in n unknowns over a commutative ring R. Prove that the equations have a solution in R if and only if for every maximal ideal of R the corresponding system in which the coefficients are replaced by their images in R_m has a solution in R_m.

6.* Consider a doubly indexed set of variables $\{x_{ij} \mid i \leq j,\ i, j \in \mathbb{N}\}$. Let S be the polynomial ring they generate over $\mathbb{C}$, so $S = \mathbb{C}[x_{11}, x_{12}, x_{22}, x_{13}, x_{23}, x_{33}, \ldots]$. For each fixed j, let P_j be the prime ideal generated by $\{x_{1j}, x_{2j}, \ldots, x_{jj}\}$. Let $U = S \setminus \bigcup_{j=1}^{\infty} P_j$. Let $R = U^{-1}S$.

1. Show that if an ideal $I \subset S$ is contained in $\bigcup_{n=1}^{\infty} P_n$, then $I \subset P_n$ for some n. [Hint: for $f \in I$, consider the (non empty, finite) set $Q(f) := \{i \in \mathbb{N} \mid f \in P_i R\}$. Show we're done unless $\forall f \in I, \exists g \in I$ such that $Q(f) \cap Q(g) = \emptyset$. Now look at $f + x_m^d g$ (which is in I) for well-chosen $m \in Q(g)$ and $d \gg 0$.]

2. Show that R has chains of primes of arbitrarily long length.

3. Prove that the localization of R at any maximal ideal is Noetherian.

4. Prove that R is Noetherian but has infinite Krull dimension. [Hint: Problem 5 on Problem Set 3 might be helpful.]