Math 614: Problem Set 5

Due Thursday December 12, 2019

1. We work in the category of R-modules (though any abelian category with enough projectives will do). A **complex** $M_\bullet$ of R modules is a sequence

$$\dots M_{n+1} \xrightarrow{d_{n+1}} M_n \xrightarrow{d_n} M_{n-1} \xrightarrow{d_{n-1}} \dots$$

such that $d_n \circ d_{n+1} = 0$ for all n. The $n-th$ **homology** of $M_\bullet$ is defined as $H_n(M_\bullet) = \frac{\ker d_n}{\mathrm{im} d_{n+1}}$. A morphism of complexes $M_\bullet \xrightarrow{\phi} N_\bullet$ is a commuting diagram

$$\begin{array}{ccccccccc} \longrightarrow & M_{n+1} & \xrightarrow{d_{n+1}} & M_n & \xrightarrow{d_n} & M_{n-1} & \xrightarrow{d_{n-1}} & \dots \\ & \downarrow \phi_{n+1} & & \downarrow \phi_n & & \downarrow \phi_{n-1} & & \\ \longrightarrow & N_{n+1} & \xrightarrow{\partial_{n+1}} & N_n & \xrightarrow{\partial_n} & N_{n-1} & \xrightarrow{\partial_{n-1}} & \dots \end{array}$$

Complexes of R-modules form an (abelian) category.

(a) Show that for every R-module M, we can construct a (possibly infinite) exact sequence $P_\bullet \twoheadrightarrow M$

$$\dots P_2 \to P_1 \to P_0 \to M \to 0$$

where each P_i is projective. The complex $P_\bullet$ is called a **projective resolution** of M.

(b) Show that any map of complexes $M_\bullet \xrightarrow{\phi} N_\bullet$ induces a well-defined map of homology $H_n(M_\bullet) \to H_n(N_\bullet)$ for each n.

(c) Two morphisms of complexes $\alpha, \beta : (M_\bullet, d) \to (N_\bullet, \partial)$ are **homotopic** if there exists a homotopy $h : M_\bullet \to N_\bullet[1]$, that is, R-module maps $h_n : M_n \to N_{n+1}$ such that

$$\alpha_n - \beta_n = \partial_{n+1} \circ h_n + h_{n-1} \circ d_n$$

for all n. Prove that if α and β are homotopic, then they defined the same map on homology.

2. For any covariant right exact functor Γ of R-modules (to another abelian category), we define the n-th **left derived functors** $L^n\Gamma(M)$ to be n-th homology of the complex $\Gamma(P_\bullet)$. For fixed R-module N, the left derived functor of the functor $- \otimes_R N$ are called "$Tor_n^R(-, N)$". That is, $Tor_n^R(M, N)$ is the n-th homology of the complex $P_\bullet \otimes_R N$.

(a) Show that

(i) $Tor_0^R(M, N) \cong N \otimes_R M$.

(ii) $Tor_n^R(M, N)) = 0$ for all $n > 0$ if N is flat.

(iii) $Tor_n^R(M, N) = 0$ for all $n > 0$ if M is projective.

(b) Let $P_\bullet$ and $Q_\bullet$ be two projective resolutions of an R-module M. Show that there are maps of complexes $\alpha : P_\bullet \otimes N \to Q_\bullet \otimes N$ and $\beta : Q_\bullet \otimes N \to P_\bullet \otimes N$ such that the composition $\alpha \circ \beta$ is homotopic to the identity map. Conclude that Tor is independent of the choice of projective resolution.

3. Define $Ext^n_R(M,N)$ to be the right derived functor of the contravariant left exact functor $Hom_R(-,N)$, that is, the n-th homology of the complex $Hom_R(P_\bullet, N)$ where $P_\bullet$ is a projective resolution of M. Show that

1. $Ext^0_R(M,N) \cong Hom_R(M,N)$.
2. $Ext^i_R(M,N) = 0$ for all $i > 0$ if N is injective
3. $Ext^i_R(M,N) = 0$for all $i > 0$ if M is projective.

4. Let $R = K[x,y,z]$ and let f be any non-constant polynomial in R and let $J \subset R$ be any non-trivial proper ideal. Show that

1. $Tor^R_i(R/\langle f\rangle, R/J) \cong Ext^i_R(R/\langle f\rangle, J) \cong 0$ for $i \geq 2$.
2. $Tor^R_0(R/\langle f\rangle, R/J) \cong \frac{R}{J+\langle f\rangle}$.
3. $Tor^R_1(R/\langle f\rangle, R/J) \cong \frac{J:f}{J}$.
4. $Ext^0_R(R/\langle f\rangle, R/J) \cong \frac{J:f}{J}$.
5. $Ext^R_1(R/\langle f\rangle, R/J) \cong \frac{R}{J+\langle f\rangle}$.

5. Let M be a finitely generated module over a Noetherian ring, and let I be its annihilator.

1. Prove that every minimal prime of I is contained in $\mathrm{Ass}(M)$.
2. Prove that $\mathrm{Supp}(M)$ is the closure of $\mathrm{Ass}(M)$.

6. Let $m_1, \ldots, m_t$ be a be monomials in a polynomial ring $R = K[x_1, \ldots, x_n]$ over a field K, and assume m_i does not divide m_j for all $i \neq j$.

1. Show that $I = \langle m_1, \ldots, m_t\rangle$ is primary iff for every x_i dividing some m_j, we have $x_i^t \in I$ for some t.
2. Find an irredundant primary decomposition of $\langle x^3, xyzw, y^2z^2, zw^2, w^4\rangle$ in $K[w,x,y,z]$. What are the associated primes? Which are minimal? Which primary components are unique?