Math 614 Daily/Weekly Update Fall 2019

This **Daily Update** will be written as soon as possible after I finish teaching at 11:30. On many days, that could mean after kids go to bed. It is basically a log of what we did, for example, by listing the relevant page numbers in the notes, but will also will correct errors and make clarifications, and provide announcements.

Tuesday Dec 10: **Thanks for a great semester!** We finished up by working out completion from the point of view of inverse limits, seeing that this produces the same result as the Cauchy sequence point of view. We extended both viewpoints to R-modules, and saw that completion in the I-adic topology defines a *right exact* functor from R-modules to R-modules. A deeper fact, whose proof uses the Artin-Rees lemma is that completion is also left exact on finitely generated modules if R is Noetherian. The worksheet is slightly rewritten, so please use the posted version.

FINAL DEADLINES: Problem Set 5 is due on Dec 12. Please print me a hard copy and leave in my door-box. Final notebooks are due Dec 20; these can be emailed as pdf if you prefer.

I am interested in your feedback on worksheets, so I can improve them for next time! Thanks!

Thursday Dec 5: We did a worksheet on the I-adic topology in R, Cauchy sequences, and completion as equivalence classes of Cauchy sequences that "have the same limit" (more precisely, which Cauchy sequences which differ by a sequence converging to zero).

Tuesday Dec 3: We covered normal Rings and the Divisor Class Group.

Tuesday Nov 26: Today's worksheet is on Krull's intersection theorem and Krull's principal ideal theorem. Everyone got through the proof of the former but not the latter. One group got stuck on 4a, so added another exercise before it to make it easier (I hope!) to compute the needed radical. Note that the new exercise uses some of the basic things we showed last time about intersections: they commutes with localization for example.

Thursday Nov 21: We worked on the *uniqueness* part of primary decomposition, which seemed to go well. Most students got through the proof.

Tuesday Nov 19: We worked on primary decomposition. I have re-written the worksheet, breaking it into two days. Today's worksheet is about existence of primary decompositions only (in a Noetherian ring). Since most students did not get to the proofs anyway, the new version should be more manageable. It includes as the last problem the complete proof of existence and uniqueness for *radical* ideals. We'll talk more about Associated primes and the Uniqueness next time.

Thursday Nov 14: We learned about Associated Primes and the Support of a module. Most students got into the proof of the finiteness of the Assassinator for finitely generated modules over Noetherian rings. Please complete the problems through number 11 at least. Problem 12 is a nice application which might be good to help practice with tensor. For sure, you need to know the statements of the proofs and many examples of various phenomena involving them.

Tuesday Nov 12: We learned about **Nakayama's Lemma,** which despite what you will see on line, has a very straightforward proof. Many applications were proved on the worksheet today, most importantly (perhaps) the equivalence of projective and free for finitely generated modules over a local ring. Most students got through this proof (Problem 9 on the worksheet) and some began thinking about the interesting example of a number ring $\mathbb{Z}[\sqrt{-5}]$. Try to finish all the problems on this worksheet by the time you turn in your final project of all worksheets. Note: the harder (starred) problems are no so hard (compared to starred problems on the homework) and can be found in Mel's notes.

Thursday Nov 7: We did a worksheet to get comfortable with exactness of some functors, and with the technique of localization to work with R-modules. For example, we saw that we could test many properties of modules and maps by testing them "locally at each prime/maximal ideal". For example, a module M is zero if and only if M_P is local for all $P \in$ Spec R (or maxSpec R), and a sequence of R-modules is exact if and only if it is exact after localization at all $P \in$ Spec R (or maxSpec R). Most students got through checking the Hom functors (for fixed source or for fixed target) are left exact, and some characterizations of projective modules. Some groups got through 5...here the statement is more important than the proof, although one should prove this once in one's career for sure.

For next time: Note that Problem Set 4 is posted. Our next topic will be Nakayama's lemma, and a deeper understanding of flat, projective and free modules. Read Mel's notes through about page 106. After that, we'll do associated primes and primary decomposition for ideals. You have a problem set due Tuesday. As for the worksheet, do not stress about the proof of Going down (complete proof available in Mel's notes). However, be sure to practice with examples and using Going Up, Going Down, Lying Over, etc.

Tuesday Nov 5: We did a "review" worksheet on tensor products. Please make sure you can complete the (unstarred) problems.

Tuesday and Thursday October 29 and 31: We did worksheets on graded rings and projective schemes. Note that I rewrote some of the problems on the Proj Worksheet to better guide you towards the proofs. Please use the newer version in writing up your solutions. We also took Quiz 7 Thursday on Valuation Rings.

Tuesday and Thursday October 22 and 24: David Speyer subbed, worksheets were done on proving that for finitely generated algebras, every prime is the intersection of the maximal ideals containing it, then on valuation rings. Quiz 6 was taken on Tuesday.

Thursday October 17: We proved some more consequences of Noether Normalization for finitely generated algebras over a field. Importantly, we saw that the dimension of a *domain* finitely generated over K is equal to the transcendence degree of its fraction field over K, which is often a good way to compute the dimension. Some students seemed to be rusty on transcendence degree. For example, some folks seemed to think that $\frac{1}{t}, \frac{1}{t-1} \in K(t)$ are algebraically independent over K. In class, I did not have time to give you an explicit polynomial showing algebraic dependence; here it is: $X - Y + XY$.

Next week, David Speyer is subbing. Please work on the Problem Set 3, and read Mel's notes through page 74. Much of this we covered earlier. You might also want to start looking at the work on power series, which we will continue with when I return, and review the stuff on bilinearity and tensor products which Mel includes (though it is also covered in 593) which goes through page 87.

Thursday October 10: We took Quiz 5. We then worked on some important consequences of Noether Normalization for **finitely generated algebras over a field.** These included the fact that a polynomial ring in d variables over a field K has dimension d. Another important fact: the dimension of a finitely generated domain over a field K is equal to the transcendence degree of its fraction field over K. The worksheet was long and (unnecessarily) hard, which I can only blame myself for. I have rewritten it, with more/better hints.

For next time: Please be sure you can solve problems through 8 on the worksheet. The order of the problems is slightly different but the first 8 are the same. Use the updated version on line, as I believe I have made your job easier on the rewrite. Problem 6 (with the ordering of the posted version) is important: the dimension of a domain finitely generated over a field K is the same as the transcendence degree of the fraction field over K. Make sure you fully understand this statement and why it is true!

Do not worry about "prime avoidance" or the stuff that comes after. This will be on the worksheet next time.

Tuesday October 8: We learned about **Noether Normalization** and some of its important applications, including (finally!) the proof of Hilbert's Nullstellensatz in the general case, and the important fact that if $R \to S$ is a homomorphism of finitely generated K-algebras, then the induced map of Spectra restricts to an induced map of maximal spectra! The proof of Noether Normalization was pretty cute, and used the uniqueness of "Base D" expansion of natural numbers.

There were some minor (mathematical) typos on the worksheet that most groups didn't notice (because they correctly interpreted the *intended* meanings of some suspect sentences). So, as always, please download and use the corrected version of the worksheet for your worksheet solutions.

For next time: Please get caught up on all worksheets. In fact, since we are about halfway through the semester soon, this is a good time to get your worksheet documents in order. In addition, please get caught up on reading Mel's notes. We are now **done** with all material through page 54 (the start of his "Oct 9" lecture). All solutions to worksheets are more-or-less in that reading. To be fully prepared for next class, read also through his next lecture (though to page 59), which will be the material for the next worksheet.

Thursday October 3: We took quiz 4. We then worked on the **going down** theorem. It is important to understand the statement and how to use it, and in particular to realize that the hypothesis are really necessary: it is easy to mistakenly convince oneself that Going Down automatically follows from Lying Over for any integral extension, but this is simply false. One group got bogged down in Problem 5 and almost convinced me there is a problem with it. However, there was an error in their thinking. I have added an extra (easy!) problem to (5) to make it easier to avoid going down that incorrect path of logic and to direct you to the correct solution.

For next time: You have a problem set due Tuesday. As for the worksheet, do not stress about the proof of Going down (complete proof available in Mel's notes). However, be sure to practice with examples and using Going Up, Going Down, Lying Over, etc.

Tuesday October 1: Sorry I forgot to give back Quiz 3! Today we worked on the Lying Over, Going Up, and Dimension theorem for **integral extensions.** Problem 9 on the worksheet has been re-written to be more clear (and correct!), so please use the edited version (up-dated on the website) when you write up solutions. Also Problem 12 has been improved.

One important point I will emphasize next time: I may have mislead you with a remark I made at the beginning of class, and want to clarify. All the theorems on Lying Over, Going Up, and the Dimension Corollary are for

integral extensions as correctly stated on the worksheet. It still makes sense, for a non-injective map $\phi : R \to S$, to ask whether it is "integral," and this turns out to be equivalent to asking that the ring **extension** im $\phi \hookrightarrow S$ is integral. This is what I meant at the beginning of class when I said "there's really no loss of generality in assuming an integral ring map is an extension:" we can replace the map $\phi : R \to S$ by the extension im $(\phi) \subset S$, and show easily that ϕ is integral if and only if the corresponding extension $\phi(R) \subset S$ is **integral extension.** But **for Lying Over, Going Up, and the Dimension Theorems/Corollaries to hold as stated, we really should have an "extension"** (as is correctly stated on the worksheet) or else apply the statements to the ring extension im$(\phi) \hookrightarrow S$.

For example, consider $\phi : R \to R/m$, which technically speaking is integral. But the map of Spec is $\{m\} \hookrightarrow$ Spec R which is **not surjective** unless Spec R consists of one point! Of course, here im$(\phi) \hookrightarrow R/m$ is the identity map, and the map of Spec for this induced integral extension is surjective.

For next time: Finish the worksheet. Use the improved version on the website for better versions of 9 and 12. Expect a quiz. Read Hochster's notes through page 48. Note that he reviews a lot of 593 material we will **assume** on the Worksheets, such as the Chinese Remainder theorem, and the division algorithm over rings for **monic** polynomials (NOTE: it doesn't hold for arbitrary polynomials, like some folks tried to say a few weeks ago). So if you are shaky on Math 593, please read carefully these sections of Hochster's notes. Next time, we will do the Going Down Theorem.

Thursday September 26: We took Quiz 3. We then did a worksheet on **commutative algebras.** For an R-algebra A, we discussed what it means to be finitely generated (a.k.a algebra-finite), module-finite, or integral. Most students got through the proof of the very important fact that *an R-algebra A is module finite if and only if it is finitely generated and integral.* The proof involves proving a "Cayley Hamilton Theorem" for commutative rings using the classical adjoint formula $B \times \text{adj}(B) = \det B \times I_n$ for any $n \times n$ matrix B over any commutative ring. If you did not finish through problem 7, please be sure to do that before next class.

For next time: Our next topic will be the behavior of the Krull dimension under integral extensions. **Please be ready** by reading Hochster's notes from Page 28 to 40. This will help you with writing up worksheets as well. Quiz 3 showed folks need practice with examples of rings! Please redo the quiz if you missed any and discuss with me if you are confused. Problem Set 2 is available to get started on, if you like.

Tuesday September 24: We had a rare lecture, just to get everyone on the same page with localization, or attempt to. We then worked at the board trying to compute the **fiber over a point** for a map of Spectra. YOUR MINIMUM HOMEWORK IS TO COMPLETE THE WORKSHEET PROBLEMS 1, 2, 3. Problem 1 is a review of Thursday's worksheet, or equivalently, of today's lecture–it's just a list of questions to help you test your understanding. Problem 3 is the proof of the Theorem on fibers, and Problem 2 is a bunch of easy practice problems computing fibers. Of course, A students will also want to practice further, problems 4 and 5 are more interesting fiber computations. Problem 6 can be skipped for now, if you are struggling to keep up (I won't put it on the quiz).

Thursday September 19: We worked on **localization** of rings, the first few problems were supposed to be review (or from the Reading; see Hochster's notes starting around page 22) but students were pretty rusty on the topic, so it took longer than I expected. Still, most students got through Problems 1-4. The core material (for now) here is Problems 1-7. One thing that slowed some folks down on the earlier problems was insistence on a particular definition of "prime ideal." In many case, these would have been more straightforward with the more concrete/basic definition using elements (i,e, P is prime if $xy \in P$ implies x or $y \in P$). Of course, a super useful and important fact (which some folks were trying to use) is that *P is prime if and only if R/P is a domain.* This point of view becomes easier to work with if we also take advantage of the universal properties of localization and quotients. To get better at this, be sure to do **Problem 7** on today's worksheet.

Problem 5 contains an important fact (that the radical of an ideal is the intersection of all primes containing it) as a corollary of the core idea in (4) describing the prime ideals in a localization RU^{-1} as the primes in R disjoint from U. Especially if you used this without proof on Quiz 1, you should make sure you can prove it!

Problem 6 also contains an important idea: Spec $(R \times S)$ can be identified with the disjoint union Spec $R\cup$ Spec S. Product rings seemed less familiar to a lot of students than I expected, so please practice by doing this exercise!

Assignment for Tuesday Sept 24: Focus on getting Problems 1-7 understood and written down. This is a good time to take stock of how your worksheet solutions are coming along. We will return to "Fibers" later...possibly as a lecture...we'll see.

Tuesday September 17: We took Quiz 2. We then worked through the proof of Hilbert's Nullstellensatz using a Worksheet in the case that K is uncountable (for example, over $\mathbb{C}$). The proof was in Problems 3-6 (though 3 was done on the last worksheet), with the ingenious trick that makes it work Problem 4b. As is often the case with the hard theorems, the hard part comes down to **clever linear algebra** (or clever combinatorics or, according to my analyst friends, clever applications of the triangle inequality).

Assignment for Thursday Sept 19: Worksheet problems 1, 3 and 7 are conceptual core material on the algebra-geometry correspondence of Commutative Algebra, whereas 4, 5, 6 are the details of the proof of the Nullstellensatz. Problems 2, 8, 9, 10 are excellent practice for working with the concepts and could be seen on a future quiz or used in future worksheet problems. Problem 11 is easy but more abstract and can be revisited later if you are overwhelmed. Next time will we discuss localization, which should be mostly review from Math 593. Read through page 28 of Hochster's Fall 2017 Math 614 notes (link provided on our Math 614 homepage) if you haven't already, and review the "universal properties" of localization and quotient rings from Math 593. I will collect Problem Set 1.

Important note about grades: The quizzes are supposed to be easy, and indeed, the Median score among folks with the prereqs was 7/8, in part because I was very picky. However, students without the prereqs did not fare as well. Please be aware that the course will be at the graduate level, and unlike "honors" courses at Michigan, grades will not be restricted to the A- to A+ range. I'm sorry that we won't be able to slow down to accommodate students who are missing Alpha course material. However, Math 593 is a great class, with a great instructor!

Thursday September 12: We took Quiz 1. We began a worksheet on Algebraic Sets. Several groups proceeded nicely through Problem 4 (or more), which is the pace I believe is right for this graduate course **if it is done carefully**. But others struggled, for two different reasons. Some folks were careful, but uncertain about basic things like how to check an ideal is maximal or what quotient rings are. These students should seriously consider Math 593, especially if they want to pass qualifying exams or go to grad school. A few others blazed through pulling in fancy concepts but missing the basic point, writing things that don't make sense or using nuclear bombs (eg, Hilbert's Nullstellensatz) to prove special cases of simple Math 412/593 facts (like a polynomial ring in one variable has a division algorithm). To these students I say: Slow down and understand more deeply!

Assignment for Tuesday Sept 17's class: Write up the (unstarred) worksheet problems through Problem 4, carefully, or through 6 or more, carefully, if you feel you fully understand and enjoy. I will return to the concepts in problems (5, 6, 7, 8) on a later worksheet in a slightly different way, so at a minimum make sure you get through 4. I know Problem Set 1 is also due next week.

THERE IS A HIGH PROBABILITY OF A SHORT (FIVE MINUTE) QUIZ NEXT TIME ON THE MOST BASIC IDEAS FROM TODAY'S WORKSHEET.

Tuesday September 10: Students did a great job going through the Worksheet on the Zariski Topology and the functor Spec. Most groups got through most of it. In writing up in your own words, the key problems to write up, at the very least, are 2, 4, 6ab, 7,11. These are the theoretical "theorems" of the class—the other problems are great for building intuition so they are also important if you really want to master commutative algebra. I strongly encourage you to sit down after class while things are fresh and write up as much as you can or as much as you are interested in. Latex is great, but handwritten is fine. Hint: you can take pictures of your work on the board.

Assignment for Thursday Sept 12's class: Read through page 28 of Hochster's Fall 2017 Math 614 notes (link provided on our Math 614 homepage). Try to focus on concretely understanding free modules, Hom, and what is **localization,** which should be review from Math 593. However, he has a lot of categorical framework around it which might make it less familiar. Start looking at Problem Set 1, which is posted on the website. I will collect your solutions next Thursday.

Tuesday and Thursday September 3 and 5: Eric Canton substituted. Tuesday he gave a brief overview on what is commutative algebra. Thursday, students worked on a worksheet on Noetherian ring. You should complete this for homework, and keep your work organized in a "Lab Notebook" that includes each worksheet with the date and your collaborators.

Assignment for Tuesday Sept 10's class: Read through page 18 of Hochster's Fall 2017 Math 614 notes (link provided on our Math 614 homepage). This includes an overview of what commutative algebra is about, and a review of the language of category theory, which will be a useful framework for us as well. [Note: Hochster's class met for 50 minute increments, so about three Hochster lectures per week is our pace, although we won't follow his course exactly.]