# Worksheet on Support and Associated Primes of Modules

Let $R$ be a commutative ring with 1. Let $M$ be an $R$ module.

DEFINITION. The **support** of $M$ is the subset $\{P \in \operatorname{Spec} R \mid M_P \neq 0\} \subset \operatorname{Spec} R$.

PROPOSITION. If $M$ is a finitely generated $R$-module, then $\operatorname{Supp}(M)$ is the closed set of Spec $R$ given by $\mathbb{V}(\operatorname{ann}_R(M))$ where $\operatorname{ann}_R M$ is the ideal $\{r \in R \mid rM = 0\}$.

DEFINITION. A prime $P \in \operatorname{Spec} R$ is an **associated prime** of $M$ if and only if there is an injective $R$-module map $R/P \hookrightarrow M$. The set of all associated primes of $M$ is called the **assassinator** of $M$ and denoted $\operatorname{Ass}(M)$.

THEOREM 1. The set of associated primes of a non-zero finitely generated module over a Noetherian ring is non-empty and finite. That is, $0 < |\operatorname{Ass}(M)| < \infty$.

DEFINITION. A **zero-divisor** on $M$ is an element $r \in R$ such that $rm = 0$ for some $m \in M \setminus \{0\}$.

THEOREM 2. Let $M$ be a finitely generated module over a Noetherian ring. The set of all zero-divisors on $M$ is the union of the Associated primes of $M$.

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(1) Let $M$ be an arbitrary $R$-module over an arbitrary ring $R$.
  (a) Show the support $M$ is empty if and only if $M = 0$. [Hint: Remember the worksheet on localization!]
  (b) Show that if $P \in \operatorname{Supp}(M)$, then $\mathbb{V}(P) \subset \operatorname{Supp}(M)$. [Hint: If $Q \supset P$, describe a natural map $M_Q \to M_P$.]
  (c) Show that the support of $R/I$ is $\mathbb{V}(I) \subset \operatorname{Spec} R$.

(2) Consider the $\mathbb{Z}$-module $M = \bigoplus_{p \text{ odd prime}} Z/p\mathbb{Z}$. Find the support of $M$ and prove it is not closed in $\operatorname{Spec} \mathbb{Z}$. Why doesn't this contradict the Proposition? [Hint: Remember $\otimes$ distributes over $\oplus$.]

(3) PROOF OF THE PROPOSITION. Let $M$ be a finitely generated module over an arbitrary $R$.
  (a) Show that $\operatorname{ann}_R M$ is an ideal of $R$.
  (b) Show that if $m_1, \ldots, m_n$ generate $M$, then $\operatorname{ann}_R(M) = \bigcap_{i=1}^n \operatorname{ann}_R(m_i)$.
  (c) Prove $\operatorname{Supp}(M) = \mathbb{V}(\operatorname{ann}_R(M))$.

(4) Let $M$ be an arbitrary $R$-module over an arbitrary ring $R$. Fix $P \in \operatorname{Spec} R$.
  (a) Show that $\in \operatorname{Ass}(M)$ if and only if $P = \operatorname{ann}_R x$ for some non-zero $x \in M$.
  (b) Show that if $R$ is a domain, the only associated prime of $R$ is $\langle 0 \rangle$.
  (c) Let $R = K[x, y]$ and let $M = R/\langle xy, x^2 \rangle$. Show that $\{\langle x \rangle, \langle x, y \rangle\} \subset \operatorname{Ass}(M)$. [Hint: Use (a). It might also be useful to remember that $K[x, y]$ is a UFD.]

(5) Show that $\operatorname{Ass}(M) \subset \operatorname{Supp} M$ for any $M$. [Hint: Use the fact that $R_P$ is a flat $R$-module, so it preserves injections.]

(6) Let $R = K[x,y]$. Fix any maximal ideal $m$ such that $R/m \cong K$. Let $M = \mathrm{Hom}_K(R, R/m)$.
  (a) Describe a natural $R$-module structure on $M$.
  (b) Show that the $R$-linear map $R \to M$ sending $r$ to the composition $R \xrightarrow{\times r} R \twoheadrightarrow R/m$ induces an embedding $R/m \hookrightarrow M$.
  (c) Show that $m$ is an associated prime of $\mathrm{Hom}_K(R, K)$.

(7) Let $R = K[x_1, x_2, x_3, \ldots]/J$ where $J = \langle x_t^{t+1} \mid t \in \mathbb{N} \rangle$. Prove that $\operatorname{Spec} R$ consists of one point and that $\operatorname{Ass} R$ is empty. Why doesn't this contradict Theorem 1? Is the reverse inclusion in Problem (5) true?

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(8) A Useful Lemma. Let $R$ be Noetherian ring and $M$ a non-zero $R$ module. Show that the set of ideals $\{J \subset R \mid \exists m \in M \setminus \{0\}\ s.t.\ J = \mathrm{ann}_R(m)\}$ has a maximal element, and any such maximal element is prime. [Hint: If $xy \in \mathrm{ann}_R\, m$, consider $\mathrm{ann}_R(xm)$. ]

(9) Proof of Theorem 2. Let $M$ be a finitely generated module over a Noetherian ring.
  (a) Let $P \in \mathrm{Ass}_R(M)$. Prove every element of $P$ is a zero-divisor on $M$. [Hint: Use (4a).]
  (b) Assume that $rm = 0$ for some non-zero $m \in M$. Show that there exists $s \in S$ such that $\mathrm{ann}_R(sm)$ is prime and contains $r$. [Hint: Use ideas from (8).]
  (c) Prove Theorem 2.

(10) Prime Cyclic Filtrations. In this problem we show that every non-zero finitely generated module $M$ over a Noetherian ring $R$ admits a filtration

$$0 = M_0 \subset M_1 \subset M_2 \subset \ldots M_{n-1} \subset M_n = M$$

such that each subquotient $M_i/M_{i-1} \cong R/P_i$ for some $P_i \in \operatorname{Spec} R$.
  (a) Use Noetherian Induction to reduce to the case that every quotient of $M$ has a prime cyclic filtration. [Hint: Recall Noetherian Induction—if we have a counterexample $M$, mod out by a submodule $N$ maximal with respect the property that $M/N$ is also a counterexample.]
  (b) Use (8) to find $x \in M$ such that $R/P \cong xR \subset M$ for some $P \in \operatorname{Spec} R$.
  (c) Prove that every finitely generated module over a Noetherian ring has a prime cyclic filtration. [Hint: Splice together $R/P$ and a filtration for $M/xR$.]

(11) Proof of Theorem 1. Fix an arbittrary ring $R$.
  (a) Prove that if $P$ is prime, then $\mathrm{Ass}(R/P) = \{P\}$.
  (b) Show that if $0 \to M_1 \to M_2 \to M_3 \to 0$ is an exact sequence of $R$-modules, then $\mathrm{Ass}(M_2) \subset \mathrm{Ass}(M_1) \cup \mathrm{Ass}(M_3)$. [Hint: If $P = \mathrm{ann}\, x$, consider two cases: either $Rx \cap M_1 = 0$ or if not. Use (a) for the second case.]
  (c) Suppose that $M_0 \subset M_1 \subset M_2 \subset \ldots M_{n-1} \subset M_n = M$. Show that $\mathrm{Ass}(M) \subset \bigcup_{i=1}^n M_i/M_{i-1}$. [Hint: Use induction on $n$ and (b).]
  (d) Prove that $\mathrm{Ass}(M) \subset \{P_1, P_2, \ldots, P_n\}$, the prime ideals appearing in a prime cyclic filtration of $M$.
  (e) Prove the Theorem on the finiteness of $\mathrm{Ass}(M)$ for Noetherian $M$ over Noetherian $R$.

(12) Let $M$ and $N$ be two finitely generated modules over a ring $R$.
  (a) Assume $(R, m)$ is local. Show that if $M$ and $N$ are non-zero, then so is $M \otimes_R N$. [Hint: Consider $M \otimes_R N \to M/mM \otimes_R N/mN \cong M \otimes_R N \cong M/mM \otimes_{R/m} N/mN$.]
  (b) Show that $(M \otimes_R N)_P \cong M_P \otimes_{R_P} N_P$.
  (c) Show that $\mathrm{Supp}(M \otimes_R N) = \mathrm{Supp}(M) \cap \mathrm{Supp}(N)$ as subsets of $\operatorname{Spec} R$.